# Generalized Mean-Field Model for Flow Control Using a Continuous Mode Interpolation

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**In the current study, a generalization of POD Galerkin models is proposed targeting strategies for experimental feedback flow control. For practical reasons, that model should incorporate a range of flow operating conditions with small number of degrees of freedom. Standard POD Galerkin models are challenged by the over-optimization at one operating condition (Deane et al. 1991). Recent successful developments to extend the dynamic range require additional modes. This leads to a control design which is less online-capable and less robust. These side constraints for control-oriented ROMs are taken into account by a 'least-dimensional' Galerkin approximation based on a novel technique for continuous mode interpolation (Morzyński et al. 2006). This interpolation allows to preserve the model dimension of a single state while covering several states by adjusting (interpolated) modes. The resulting 3-dimensional Galerkin model is presented for the transient flow around the circular cylinder and shown to be in good agreement with the corresponding direct numerical simulation. The 3-dimensional model is based on the shift mode and interpolates between the two most energetic POD modes of natural limit cycle and the complex stability eigenmode representing the fixed-point dynamics. The interpolation technique is also used to approximate the POD modes with suitable stability eigenmodes. These efforts shall lead to *a priori* models from first principles without the need for empirical flow data.**

## I. Introduction

Closed-loop flow control is characterized by a very short time between sensing of flow properties and adequate actuating action. Depending on the flow conditions the time required for the action is of order of milliseconds. It is obvious that under such circumstances only the Reduced Order Models (ROMs) can be

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used in the controller design. The extensive survey of the model-based control can be found, for example, in.[1,2] Proper Orthogonal Decomposition (POD) Galerkin model, the most frequent tool for control design, can exhibit dynamic fragility even for natural flow conditions. The simplest remedy for this fact is to increase the number of degrees of freedom of the model on cost of additional computational effort. Much more improvement of Galerkin model can be gained by combining two state descriptions as demonstrated by Noack et al. in.[3] That Galerkin model was build on POD resolving the attractor and stability eigenmodes resolving the linearized dynamics. Thus, transient and post-transient flow behavior was accurately predicted. The concept of hybrid model reduces significantly the number of necessary degrees of freedom of the system. However, this approach is based on a union of modes. In similar spirit, Ma et al. in[4] add modes of 2D vortex shedding to 20 POD of the 3D flow to resolve the transition from 2D to 3D wake dynamics.

The narrow dynamic bandwidth of POD-based Galerkin model is even more pronounced for controlled flow and under changing operating conditions. One example of operating condition is the Reynolds number. The deviation from the Reynolds number for which the Galerkin approximation is made results in poor properties of the Galerkin model. Deane et. al.[5] showed that the POD modes determined for $Re = 100$ can resolve only 35% of the fluctuation energy at $Re = 150$. Preserving adequate dynamics for model-based MISO control with the use of Galerkin flow model[6] necessities about 40 POD modes obtained of transient forced data while only 2 POD modes are required for natural flow.

The "ideal" set of modes should not only adequately represent the flow dynamics and transients but also "adapt itself" to the controlled flow conditions. The traditional Galerkin models do not include the mode adjustments on operating condition. The idea of mode parameterization instead of increasing the number of POD modes is pursued by several groups. Luchtenburg et al.[7] employ a parameterization based on control. Further improvement of the Galerkin model for control purposes is presented in.[8] The POD modes are derived from the controlled flow subjected to moderate to aggressive forcing. In this way the dominant modes change along the transients is taken into account. The concept of interpolated POD modes used for circular cylinder flow control is elaborated in details in.[8] While retaining relatively large number of modes - about 40 - only 3 Fourier coefficients have to be dynamically estimated in the controller. Siegel et al.[9] employ short time sampling of transient flows to improve the dynamical properties of the model.

In this paper a novel continuous mode interpolation technique is proposed. The mode interpolation smoothly connects not only different operating conditions, but also stability and POD modes. In addition, the extrapolation of modes outside the design conditions is illustrated. In the final section it is shown how interpolated modes enable 'least-order' Galerkin models keeping the dimension from a single operating condition but resolving several states. These models are especially well suited for control design.

To develop the proposed methodology the benchmark problem must be stated. One of the most common benchmark problems for methodological developments of the controller is the bluff body wake dynamics with special emphasis to flow about a circular cylinder. Since von Kármán model (1911), numerous new models for this flow have been proposed but only some are exploited for model based control design.[10,11] The suppressing of vortex shedding can be a challenging benchmark for closed-loop control with active means.

## II. Different operating conditions

It is of interest of control design to smoothly and continously pass from one operating condition to another. For example for circular cylinder operating conditions range from the state near the fixed point (steady solution) to the limit cycle (time-averaged flow) (figure 2).

While the fixed point operating conditions are precisely described by flow stability eigenmodes with the transition and increased value of the disturbance amplitude the modes structure changes up to the one corresponding to oscillatory limit cycle related to the POD modes. In the same time the coherent structures are distorted, the oscillation period and growth rate increases, and the maximum of the turbulent kinetic energy moves toward the cylinder (figure 4). All these phenomena cannot be accounted by simple linear combination of both mode kinds as it leads directly to the beat phenomenon instead of desired interpolated value. In the controlled flow situation the use of POD modes computed for the wrong orbit results in false prediction of the phase of the controller and leads to deterioration of the controlling effect.

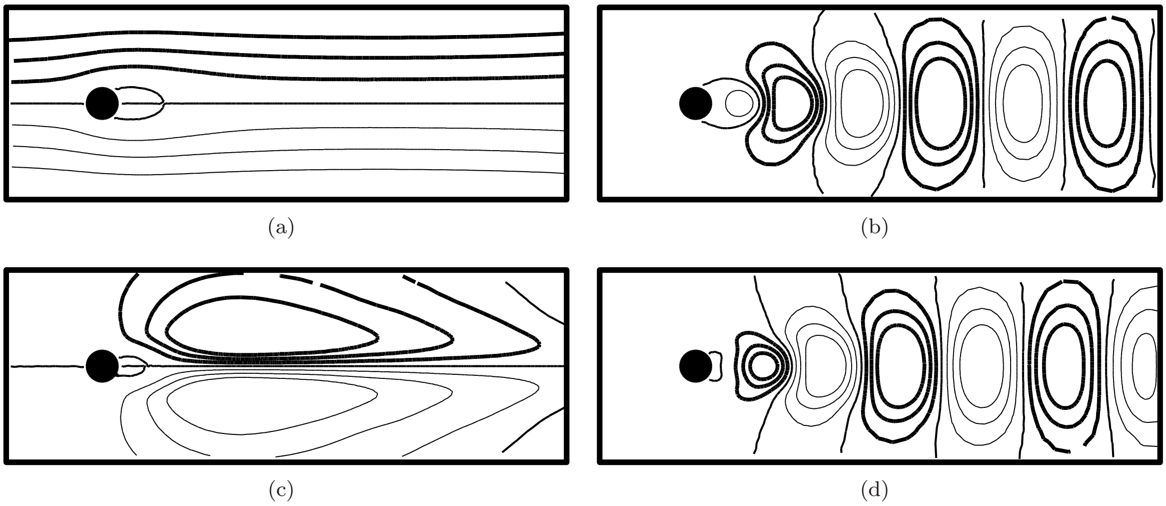



**Figure 1. Streamlines of the mean flow (a), shift mode (c), and first POD modes (b), (d)**

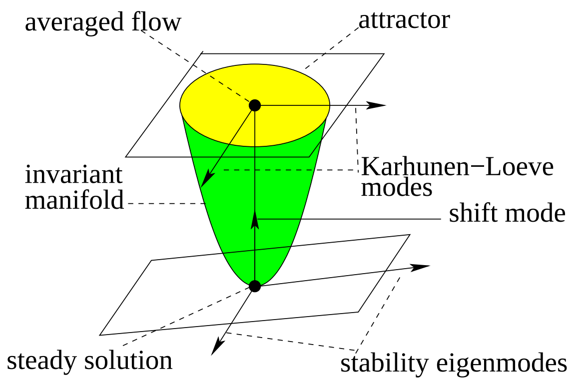



**Figure 2. Principle sketch of the generalized mean-field paraboloid**

## III. Interpolated modes

To overcome the problem of inadequate modes and preserving the low dimensional character of the model several techniques [9] were proposed.

Here the novel approach is presented. We target the least-dimensional models by mode interpolation, i.e. always being in minimum subspaces.

Flow control design needs continuous interpolation between different states. In particular interpolation between stability eigenmode basis, POD mode basis and stability-POD basis must be performed. For any discretization of the computational domain the matrix eigenvalue problem (resulting from flow stability analysis and POD) has the same dimension at all operating conditions. In the present study FEM is applied for discretization. A linear interpolation of the underlying eigenproblems can be performed with $\kappa$ as the interpolation parameter. Here, $\kappa = 0$ for the steady flow and stability eigenmodes and $\kappa = 1$ for the periodic flow and POD modes.

The simplest form of local linearization can be obtained by linear interpolation of matrices $A_o$ and $A_s$ at two different states:

$$A(\kappa) = A_s + \kappa(A_o - A_s) \tag{1}$$

The $A_s$ matrix refers to the steady flow conditions and $A_o$ to the periodic limit cycle.

In practice this technique is adequate only for mode interpolation of the same basis. The stability matrix with non-normal eigenvectors and complex eigenvalues requires special treatment to be interpolated with symmetric Fredholm kernel, hermitian eigenvalue problem with real eigenvalues. Details of this procedure can be found in.[12]

# IV. Reduced Order Models for control and toward *a priori* model of the flow

In the current study, we consider two-dimensional incompressible flows around stationary cylinders with size $D$ placed in a uniform stream with velocity $U$. The velocity and pressure field are denoted by $\mathbf{u}$ and $p$. The flow is characterized by the Reynolds number $Re = UD/\nu$, $\nu$ being the kinematic viscosity of the fluid. The non-dimensionalized Navier-Stokes equation of incompressible flow reads

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \frac{1}{Re} \triangle \mathbf{u} + \sum_{l=1}^{N_b} b_l(t)\, \mathbf{g}_l(\mathbf{x}). \tag{2}$$

The classical[13–15] Galerkin approximation of a velocity field $\mathbf{u}(\mathbf{x}, t)$ is given by:

$$\mathbf{u}(\mathbf{x}, t) = \sum_{i=0}^{N} a_i(t)\, \mathbf{u}_i(\mathbf{x}), \tag{3}$$

where $a_0 \equiv 1$ for reasons of simplicity.

The generalized Galerkin method based on interpolated modes will differ from the standard one due to the presence of $\kappa$ in the model.

This gives rise to a generalized Galerkin approximation

$$\mathbf{u}^\kappa(\mathbf{x}, t) = \sum_{i=0}^{N} a_i^\kappa(t)\, \mathbf{u}_i^\kappa(\mathbf{x}). \tag{4}$$

Again, $a_0^\kappa \equiv 1$. The superscript $\kappa$ refers to the operating condition which this expansion shall approximate.

Traditional Galerkin models do not include the mode adjustments for time-varying operating condition. However, modes varying with stationary operating conditions have been utilized before.[16,17]

The resulting low-dimensional Galerkin model can be derived from Eq. (2) and Eq. (4) in a straightforward manner[18] assuming a slowly varying $\kappa$,

$$\frac{d}{dt} a_i^\kappa = \frac{1}{Re} \sum_{j=0}^{N} l_{ij}^\kappa a_j^\kappa + \sum_{j,k=0}^{N} q_{ijk}^\kappa a_j^\kappa a_k^\kappa + \sum_{l=1}^{N_b} g_{il}^\kappa b_l, \tag{5a}$$

$$\frac{d}{dt} \kappa = F(\kappa, \mathbf{a}^\kappa, \mathbf{b}). \tag{5b}$$

The Galerkin system coefficients $l_{ij}^\kappa$ and $q_{ijk}^\kappa$ depend on the operating condition $\kappa$. During a natural transient of an oscillatory flow, $\mathbf{u}_0^\kappa$ may be the phase-averaged velocity and $\kappa$ may be identified with the shift-mode amplitude.[3] Moreover, Eq. ($5b$) is derived from the Navier-Stokes equation filtered with the phase average $\langle \quad \rangle$ and projected onto the mean-field correction. During an actuated transient, $\kappa$ may be identified with a forcing amplitude or frequency. In this case, Eq. ($5b$) is replaced by a prescribed dynamics of $\kappa$.

In a simple case, the Galerkin system coefficients may be linearly interpolated between two close operating conditions identified by $\kappa = 0$ and $\kappa = 1$:

$$l_{ij}^\kappa = l_{ij}^0 + \kappa\, (l_{ij}^1 - l_{ij}^0), \quad q_{ijk}^\kappa = q_{ijk}^0 + \kappa\, (q_{ijk}^1 - q_{ijk}^0). \tag{6}$$

It should be noticed that, targeting least-dimensional representations, the whole model is reduced to 4 ODEs. In the method presented here only two interpolated modes, shift mode and the parameter $\kappa$ represent the flow in the very wide range of significantly different operating conditions. From the point of view of the flow control practice this is of the highest importance.

In the same time the proposed method of ROM of the flow can be a crucial step toward construction of the fully *a priori* model of the time-averaged flow, based only on the steady solution of the Navier-Stokes equations and physical stability eigenmodes.

## V. Interpolation between POD and stability eigenmode basis

The method presented in section §III is tested on interpolation of POD and stability eigenmode basis. This example represents a more complex problem than simple linear interpolation between basis of the same mode and is also more important for practical applications.

Figure 3, 4 shows the results of the corresponding interpolated modes $\mathbf{u}_i{}^{\kappa}$. The detail results can be found in.[12]

For interpolation the real part of the most unstable stability mode and the first POD mode is applied. Interpolation, performed for the wake behind a circular cylinder at $Re = 100$ is shown in figure 3. In figure 4 the energy $q = \frac{1}{2}\left\|\mathbf{u}\right\|^2$ on the centerline is depicted. The modes change smoothly from one operating state to another. Thereby, the maximum of fluctuation level moves to the outflow. A similar observation has been made by Lehmann et al.[8] for the flow subjected to increasingly aggressive stabilizing control.

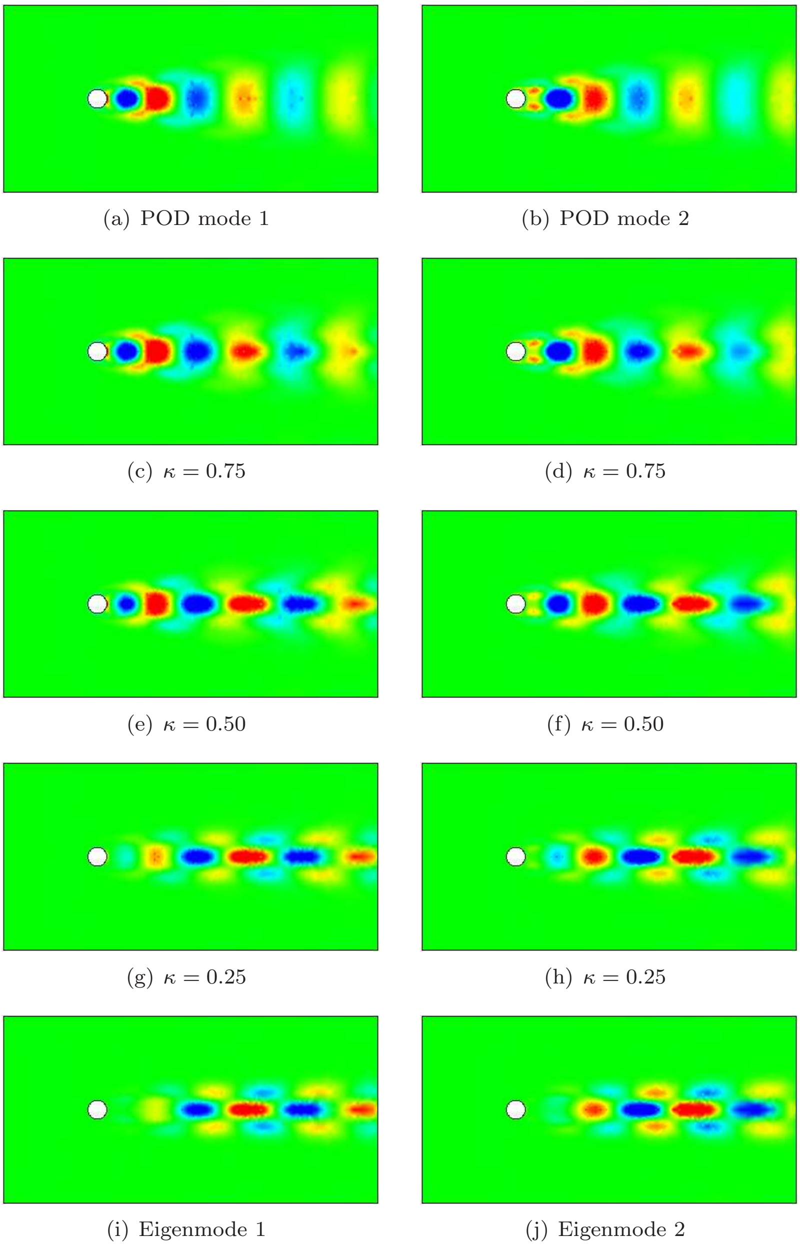



**Figure 3. Interpolated modes. Bottom: real (left) and imaginary part (right) of the dominant eigenvector. Top: Two most energetic POD modes. Middle: Interpolated modes for $\kappa$=0.75, $\kappa$=0.50 and $\kappa$=0.25. In the figure, the vorticity of the modes is depicted.**

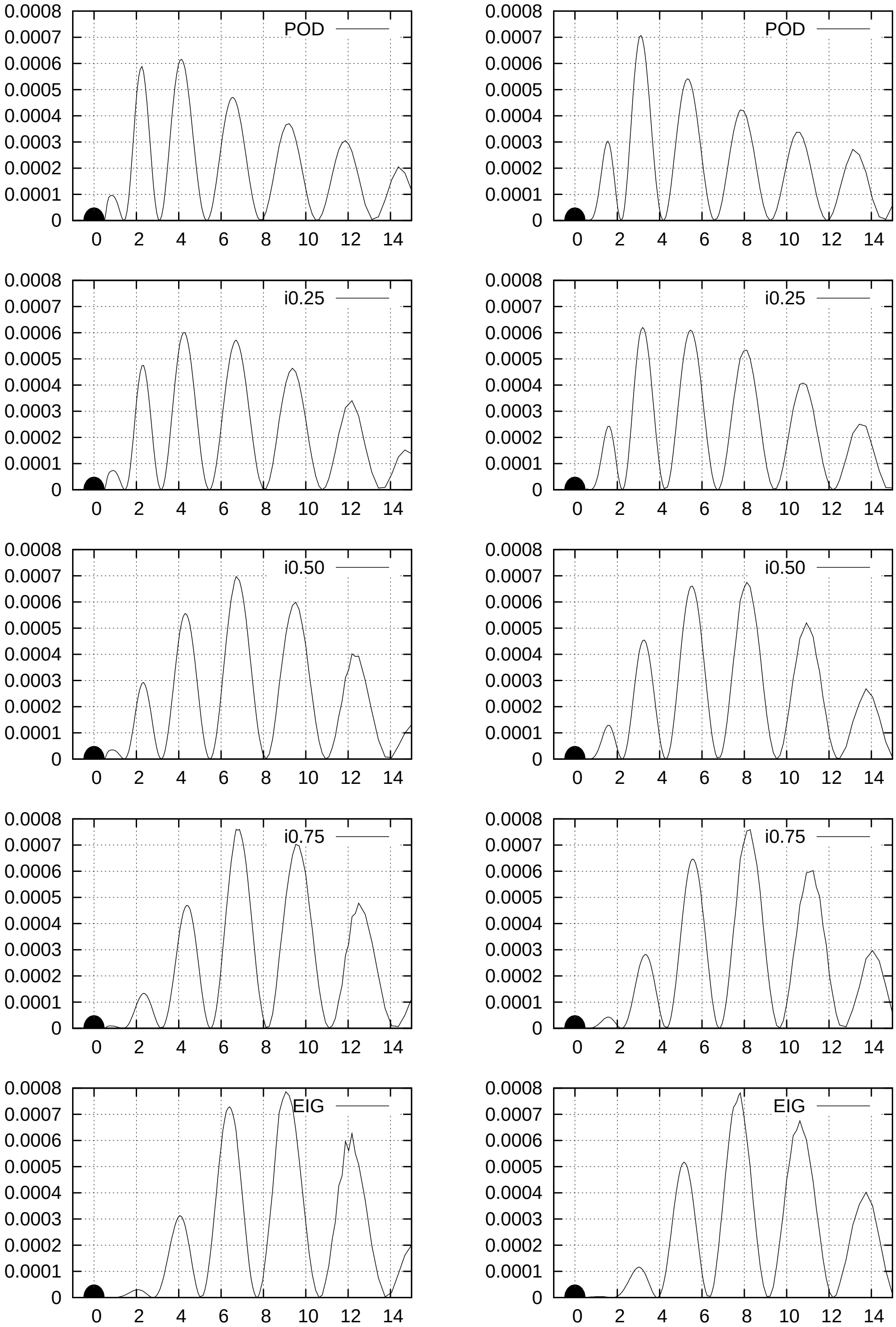
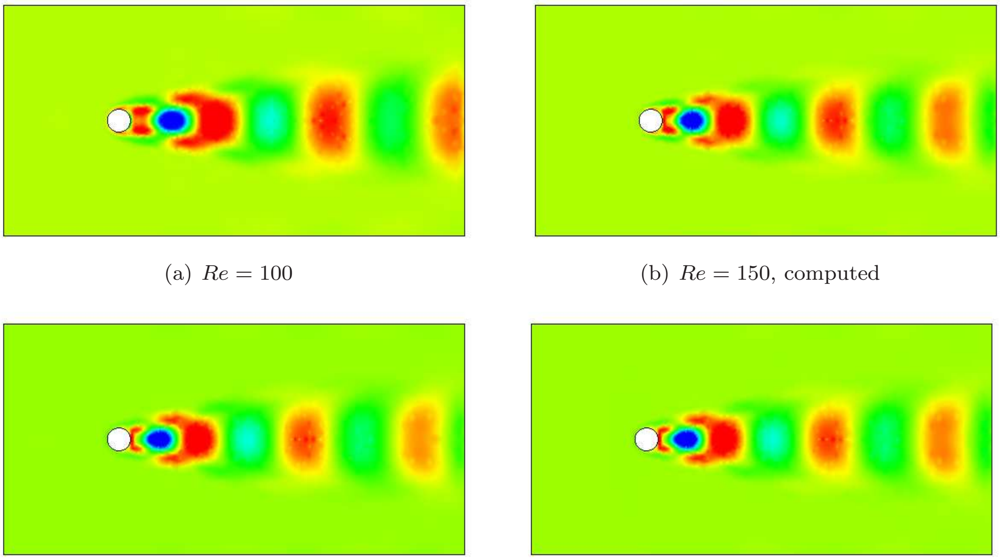



**Figure 4.** $q = \frac{1}{2}\mathbf{u}^2$ **distribution along x-axis for first two modes. Top: Two most energetic POD modes. Bottom: Real and Imaginary part of the dominant eigenvector. Middle: Interpolated modes for $\kappa$=0.75, $\kappa$=0.50 and $\kappa$=0.25.**

# VI. Mode extrapolation for other operating conditions

In many practical situations the estimation of the unknown flow state based on information already available is of great interest. The flow control design is the natural candidate to use any application solving this problem. Also CFD depends on *any* "guess solution" estimation dramatically changing the iteration convergence. Experiments with continuous mode interpolation, shown in section §V are here ground for further investigation of mode extrapolation.

Interpolation shown in figure 3 and 4 rises the question if it is possible to *extrapolate* the operating conditions by setting $\kappa > 1$ (or $\kappa < 0$). In this way it would be possible to determine the approximation of the flow field for new conditions *without* computations. This attractive feature of the approach presented here is also tested. The POD modes for $Re = 100$ and $Re = 125$ were applied to predict the conditions at higher Reynolds number, $Re = 150$. The result is shown in figure 5.



(a) $Re = 100$ (b) $Re = 150$, computed

(c) $Re = 125$ (d) $Re = 150$, extrapolated

**Figure 5. Approximation of POD mode for higher Reynolds number basing on extrapolation between POD modes at the lower one.**

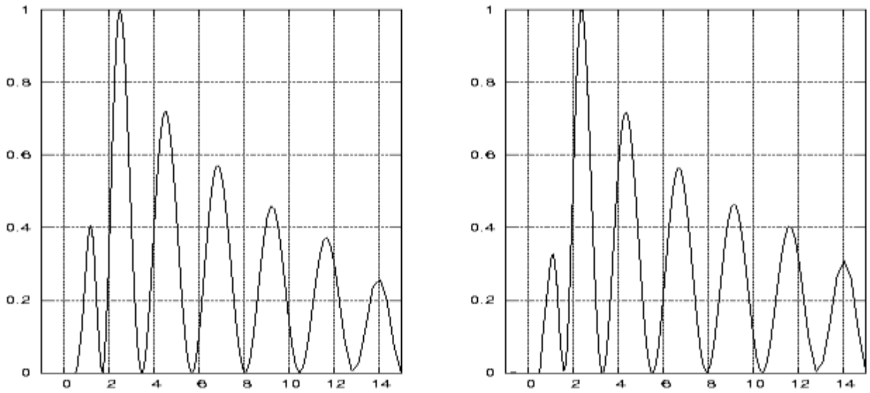


**Figure 6.** $q = \frac{1}{2}\mathbf{u}^2$ **distribution along x-axis for computed (left) and extrapolated (right) POD mode.**

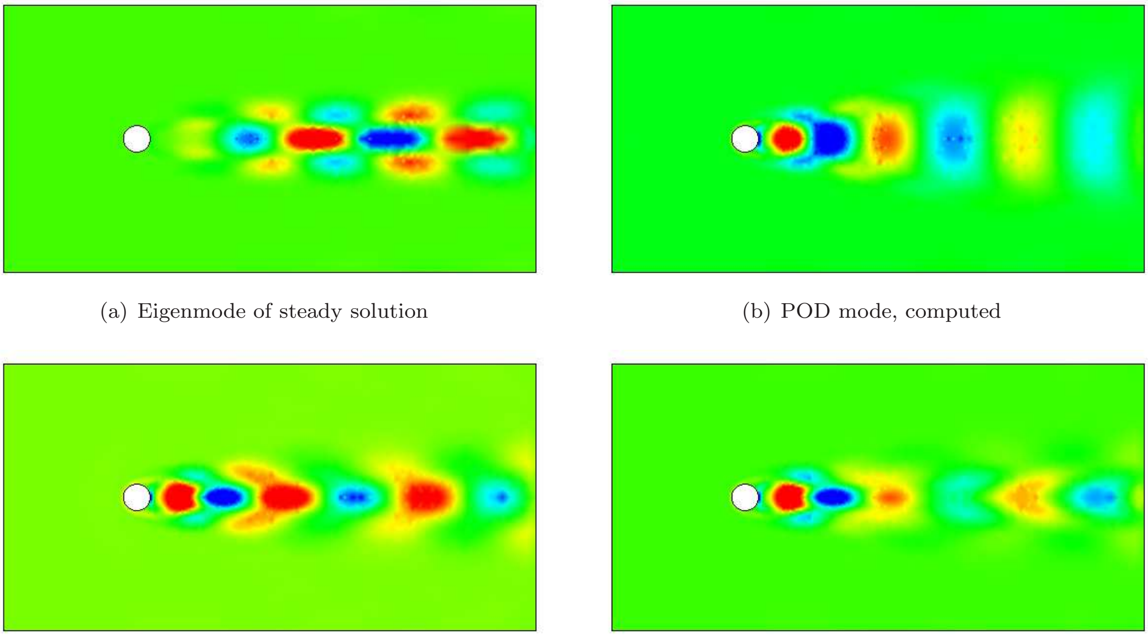
The right hand side of the Figure shows almost indistinguishable vorticity fields for the computed POD mode and the extrapolated one. The closer look to the fluctuation energy plot in figure 6 shows minimal differences between the latter two. In[12] it is shown that extrapolation for smaller Reynolds number performs equally well.

## VII. Approximation of a POD basis using stability eigenmodes

The ultimate goal of modeling time-averaged flow is limitation of empirical input to a minimum. A major step towards this model would be prediction of POD modes on base of stability eigenmodes. Global flow stability analysis predicts exactely the periodic flow behaviour near the fixed point, only with the steady base flow. Composition of this predictive feature with continuous mode interpolation should accomplish the desired solution.

Here extrapolation of physical modes to approximate empirical ones is attempted. Starting points are eigenmodes computed from a global stability analysis of the steady solution and of time-averaged flow. The POD mode shall be expressed in terms of these physical modes (stability eigenmodes) only.

The real parts of the most unstable eigenmodes at $Re = 100$ are depicted in left part of figure 7. With these stability modes the POD-mode approximation is constructed. The result is shown in right bottom part of figure 7 and compared with the POD mode at $Re = 100$ computed from the snapshots of unsteady flow simulation. In the near wake, the extrapolated field is in very good agreement with the POD computation. As the near wake is particularly important for flow control, the results of the *a priori* mode interpolation are encouraging.



(a) Eigenmode of steady solution

(b) POD mode, computed

(c) Eigenmode of mean flow

(d) POD mode, approximated ($\kappa = 1.44$)

**Figure 7. Approximation of POD mode extrapolating from two eigenmodes at $Re = 100$. For details see text.**

# VIII. Galerkin model based on interpolated modes

In this section, the Galerkin model based on continously interpolated modes (described in section §IV) is presented and compared with commonly used POD Galerkin models and DNS.

To assure better adjustment for time-varying operating conditions, two mode sets are included. In each set, base flow and shift mode remain unchanged, while two remaining modes depend on operating conditions. For the state corresponding to the fixed point, the two most unstable eigenmodes of global stability analysis are used. For limit cycle dynamics, two first POD modes have been chosen.

For both mode sets the Galerkin projection have been performed, leading to two sets of Galerkin system coefficients $l_{ij}^{\kappa}$ and $q_{ijk}^{\kappa}$, where $\kappa = 0$ represents fixed point (eigenmodes) and $\kappa = 1$ represents limit cycle (POD modes).

In proposed Galerkin model, constructed as shown in Eq. (5), the interpolation parameter $\kappa$ is identified with shift-mode amplitude $a_3$:

$$\kappa = \frac{a_3}{a_3^\bullet} - 1, \tag{7}$$

where $a_3^\bullet$ represents amplitude of the shift mode for periodic solution.

This choice assures that during the transition of the flow from steady state into periodic one $\kappa$ changes continously from 0 to 1 (figure 8), allowing proper interpolation of used modes and Galerkin system coefficients given by Eq. (6).

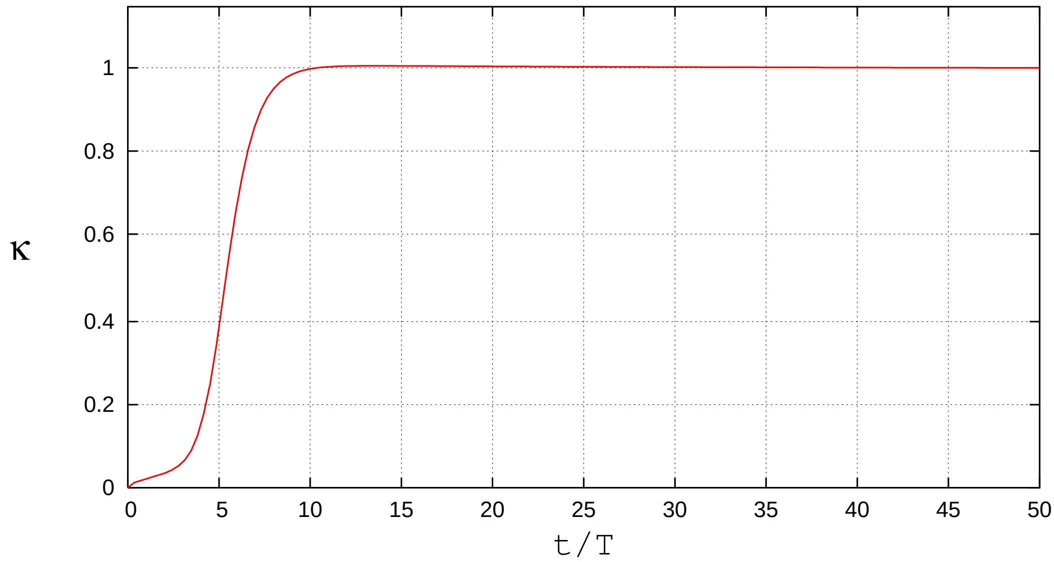



**Figure 8. Interpolation parameter $\kappa$ in dependency of the time for the transient cylinder wake. This time is normalized by the period $T$. For details see text.**

Since the interpolated modes are matching the transients of the flow better than POD modes, the dynamical behaviour of proposed model (represented in figure 9 by the turbulent kinetic energy of fluctuation $K$) is much closer to DNS results than in case of POD-only models.

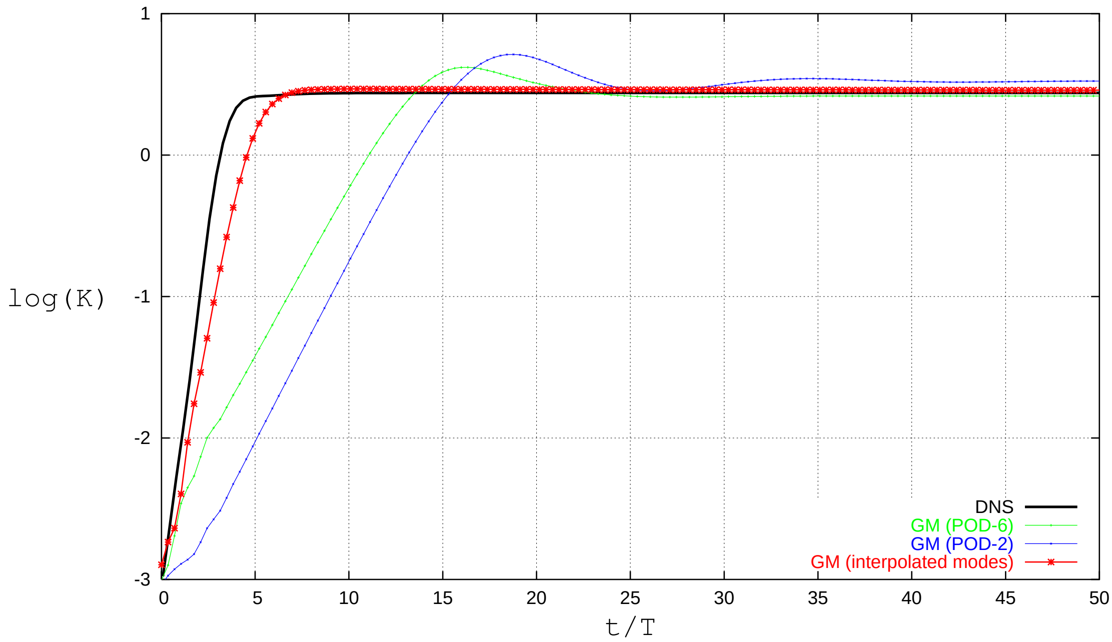



**Figure 9. Turbulent kinetic energy of the fluctuation around mean-flow in dependency of time normalized by the period $T$. Transient solutions for DNS (thick black line), Galerkin models based on 2 POD modes (blue line), 6 POD modes (green line) and GM including continuous mode interpolation (red line).**

The transient time is comparable to the one obtained in the Navier-Stokes simulation, and about three times shorter than in POD-model cases. The over–estimation of $K$ is almost absent and its final value is much closer to DNS than in other cases.

Summarizing, the presented model, containing only 3 degrees of freedom (two interpolated modes and shift-mode), is more accurate than 7-dimensional POD model.

# IX. Conclusions

In the current study, a key enabler for control-oriented POD Galerkin models is proposed: a novel continuous mode interpolation.[12] Thus, a 3-dimensional Galerkin model is constructed which describes robustly and accurately the transient flow around a circular cylinder from the steady solution to periodic flow. The underlying Galerkin approximation resolves the mean-field correction[19,20] via the shift-mode and the oscillatory fluctuations by an interpolation between stability eigenmodes and POD modes. Thus, an efficient generalization of mean-field models at large supercritical Reynolds numbers is proposed.

The incorporation of small flow changes in interpolated modes allows to preserve model dimension of a single state while extending the dynamic range to other states included in that interpolation. Related mode-interpolation studies are based on new simulations for the intermediate states[7] or on transient flow data,[6,21] i.e. require information beyond the two reference conditions.

It should be noted that the very ansatz of continuous mode interpolation allows to construct structures which have not been computed before. Examples for the wing at different angles of attack and a Ahmed body at different distances from the ground are provided by in.[12]

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