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Three-dimensional stability analysis of the periodic flow around a circular cylinder

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The onset of three-dimensionality in the von Kármán vortex street behind a circular cylinder is investigated by carrying out the first global, nonparallel, three-dimensional stability analysis of the periodic flow. This flow is found to become unstable at a Reynolds number of 170 by a critical, three-dimensional Floquet mode with a spanwise wavelength of 1.8 diam. The spatial structure of this mode indicates that the onset of three-dimensionality is due to a near-wake instability and not caused by a stagnation-line or a boundary-layer instability.

Experimental[1,2] and numerical investigations[3] indicate that the periodic two-dimensional flow around a circular cylinder becomes unstable by three-dimensional perturbations which are periodic in spanwise direction. The observed critical Reynolds numbers range from 150 (Ref. 4) to 200 (Ref. 5). Unfortunately, there exist no published experimental or theoretical quantitative investigations about the spanwise wavelength. The physical origin behind this instability has not been elucidated so far. The speculations about this origin include a stagnation-line, a boundary-layer, and a near-wake instability. Yet, the clarification of the physical processes involved in the transition is of great interest for approximate analytical theories and for local stability investigations.

This clarification can be achieved by a global, three-dimensional stability analysis of the periodic flow. For this purpose, the two-dimensional Galerkin method[6,7] has been generalized to three dimensions[8,9] employing Zebib's[10] transformation of the Navier–Stokes equation [see his Eqs. (4)–(7)]. Here, the incompressible velocity field $\mathbf{u}$ in terms of the cylindrical coordinates r, ϕ, z, and the time t is expressed by $\mathbf{u}=\nabla\times\Psi\mathbf{e}_z+\nabla\times\nabla\times\Phi\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in spanwise direction and Ψ, Φ can be considered as "generalized streamfunctions." In the present Galerkin method, these functions are approximated by the finite expansions $\Psi=R^{(0)}(r)\sin\phi+\Sigma_{k=-1}^{0}\Sigma_{j=-4}^{4}\Sigma_{i=0}^{6}\,a_{ijk}^{(1)}(t)\times R_i^{(1)}(r)\Phi_j(\phi)\,Z_k(z)$ and $\Phi=+\Sigma_{k=1}^{1}\Sigma_{j=-4}^{4}\Sigma_{i=0}^{6}\,a_{ijk}^{(2)}(t)\times R_i^{(2)}(r)\,\Phi_j(\phi)\,Z_k(z)$. The radial modes $R^{(0)}$, $R_i^{(\kappa)}$ ($\kappa=1,2$) and the azimuthal modes Φ_j are constructed so that the resulting velocity field satisfies the boundary conditions at the cylinder and at infinity for all choices of the 189 Fourier coefficients $a_{ijk}^{(\kappa)}$. The spanwise modes $Z_{-1}=(1/\sqrt{\pi})\cos(k_z z)$, $Z_0=1/\sqrt{2\pi}$, $Z_1=(1/\sqrt{\pi})\sin(k_z z)$ induce a periodicity in spanwise direction with the wavelength L and the wave number $k_z=2\pi/L$. Detailed descriptions of the Galerkin method are provided in Refs. 6–9.

The Floquet stability analysis of the two-dimensional, periodic flow is based on the numerical integration of the linearized evolution equations. The computation of the Floquet multipliers μ_m and the corresponding Floquet modes follows the traditional guidelines.[6] It should be noted that there always exists a Floquet multiplier $+1$, which corresponds to a phase shift on the limit cycle. This multiplier is neglected in the sequel, since the phase shift is immaterial for the Poincaré stability properties of the limit cycle.

Figure 1 displays Floquet spectra for two Reynolds numbers and three wave numbers $k_z=2\pi/L$. All Floquet multipliers which are associated with two-dimensional Floquet modes lie within the unit circle, i.e., the periodic flow is asymptotically stable with respect to two-dimensional disturbances. This stability property is confirmed for all considered Reynolds numbers from the onset of the periodicity at $\mathrm{Re}\gtrsim 50$ up to 500. Similarly, the three-dimensional Floquet multipliers are seen to be almost in the origin for sufficiently large wave numbers. In other words, three-dimensional disturbances with small spanwise structures are dissipated within one shedding period. For large spanwise wavelengths, however, there exist no similar dissipation processes and the periodic flow is seen to be neutrally stable—due to three-dimensional Floquet multiplier near $+1$. This neutral stability, which is evidenced for all regular and transitional Reynolds numbers, explains the experimental observation of laminar vortex formations with large spanwise structures, like oblique vortex shredding, chevron patterns, etc.[11–13]

The instability occurs first at the critical wave number of $k_{z,\mathrm{crit}}=(1.75\pm0.01)/R$ and the critical Reynolds number of $\mathrm{Re}_{\mathrm{crit}}=170\pm1$, when the positive three-dimensional

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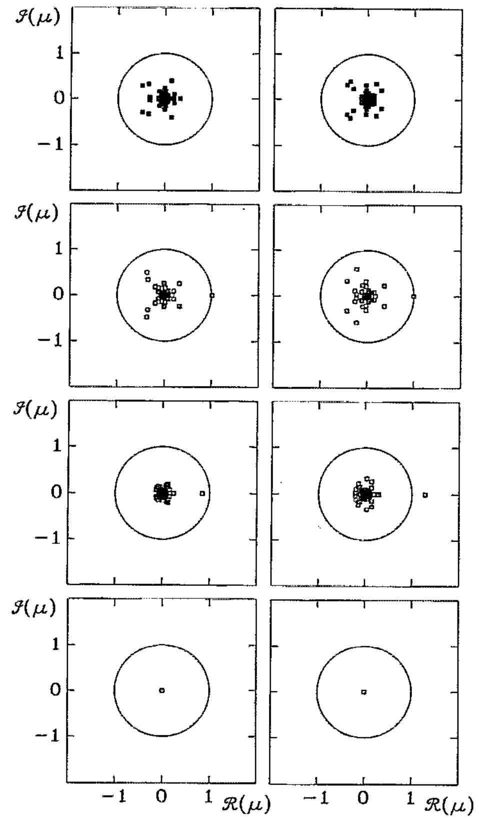

FIG. 1. Floquet multiplier configurations in the complex plane, for Re $=150$ (left column), 200 (right column) and the two-dimensional stability analysis (first row), $k_z=0.1/R$ (second row), $k_z=1.75/R$ (third row), $k_z=4.0/R$ (fourth row). Each solid square represents one two-dimensional Floquet multiplier μ_m which is, by definition, independent of the spanwise wave number. Open squares denote k_z-dependent three-dimensional multipliers μ_m. $\Re(\mu)$ and $\Im(\mu)$ denote the real and imaginary axis, respectively.

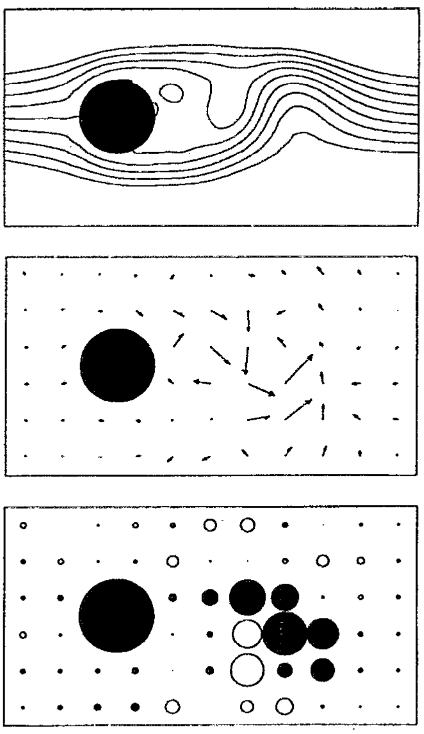

FIG. 2. Two-dimensional periodic solution (top) and three-dimensional Floquet mode (middle, bottom) for Re=200 and $k_z=2\pi/L=1.75/R$ at the same instant. The periodic flow is visualized with streamlines. The velocity flow of the Floquet mode is shown in planes with constant spanwise coordinate $z=0$ (middle) and $z=L/4$ (bottom). The large solid circle represents the cylinder. The arrows illustrate the magnitude and direction of the tangential velocity component. The diameter of a circle in the lower picture is proportional to the absolute value of the velocity component in spanwise direction; its sign is positive (negative) for solid (open) circle.

Floquet multiplier leaves the unit circle (see Fig. 1). This critical point has been determined by carrying out 36 Floquet analyses for $\mathrm{Re}=150,160,...,200$ and $k_z=1.5,1.6,...,2.0$ and interpolating the position of the critical multiplier for $150<\mathrm{Re}<200$ and $1.5<k_z<2$.

The corresponding Floquet mode is displayed in Fig. 2. This mode is normalized to yield similar spanwise velocity amplitudes as the asymptotic three-dimensional solution for the same wavelength and Reynolds number. The largest spanwise velocity components are concentrated in the near-wake region, where the streamlines have the largest curvature. The resulting velocity field of the Floquet mode superimposed on the periodic solution displays two counter-rotating vortices in downstream direction per spanwise wavelength. After half a period, these vortices occur at the other side of the cylinder with opposite orientation. In the boundary layer, no spanwise flows are observed within the accuracy of the employed Galerkin method. Hence, the Floquet analysis indicates that the onset of three-dimensionality in the periodic cylinder wake is caused by a three-dimensional near-wake instability. This three-dimensional instability may also be excited at subcritical Reynolds numbers by a periodic acoustic field.[14]

The dependency of the critical Floquet multiplier μ_{crit} (with the largest modulus) on the Reynolds number and the wave number is illustrated in Fig. 3. Clearly, the periodic flow is seen to be neutrally stable for $k_z\to 0$. For large k_z, $|\mu_{\mathrm{crit}}|$ is constant since this quantity is associated with a k_z-independent two-dimensional Floquet mode. At $k_z \approx k_{z,\mathrm{crit}}=1.75/R$, the amplification rate $|\mu_{\mathrm{crit}}|$ displays a sharp peak, thus indicating that the characteristic spanwise wavelength is $L_{\mathrm{crit}}\approx 3.6R=1.8D$, where R and D represents the cylinder radius and diameter, respectively. This length is very pronounced and can be seen in our experi-

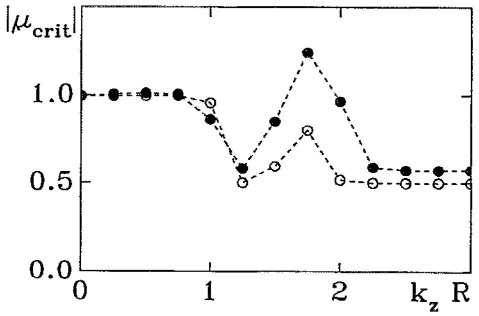

FIG. 3. Spectral radius $|\mu_{\mathrm{crit}}|=\max_{m=1,...,N}|\mu_m|$ of the three-dimensional Floquet spectrum in terms of the wave number k_z. The open (solid) symbols represent the values for Re=150(200).

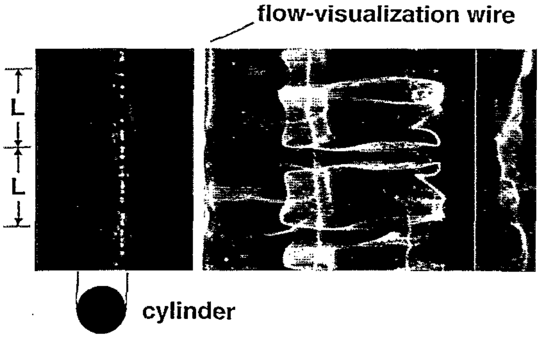

FIG. 4. Experimental streak surface for Re=200. The flow direction is from left to right. The flow-visualization wire is positioned parallel to the cylinder axis in the near wake. For further details of the experimental setup see Ref. 13.

mental flow visualizations (see Fig. 4) obtained in a water towing tank with a hydrogen bubble technique. Based on these results, one would expect that it is very difficult to "excite" other spanwise wavelengths by external perturbations. These expectations are experimentally confirmed by Sheridan's group[15] in Clayton, Australia.

The obtained spanwise wavelength with maximal amplification, $L_{crit}=1.8D$, agrees well with the corresponding value of $L=\pi R\approx 1.6D$ chosen by Karniadakis and Triantafyllou[3] and by Tomboulides *et al.*[16] for their numerical simulation of the three-dimensional cylinder wake. Experimentally, a wavelength of approximately $L=1.7D$ is obtained for a Reynolds number of 200 (see Fig. 4). In Fig. 4, each pair of tangles with hydrogen bubbles corresponds to one pair of oppositely oriented vortices in downward direction, i.e., one spanwise wavelength. (Some experimentalists take the distance between two neighboring tangles as a spanwise wavelength, thereby obtaining half the theoretical value.) It should be noted that the spanwise wavelength with maximal amplification slightly decreases with increasing Reynolds number.[15] Hence, the experimental wavelength for Re=200 is somewhat lower than the theoretical one for Re=170. Since Karniadakis, Triantafyllou, and Tomboulides' numerical simulations are also carried out for Re$\gg$200, they obtain better results with a slightly lower wavelength than the onset value of the present stability analysis. Taking this into account, theoretical and experimental results are in good agreement.

It can be concluded that the two-dimensional periodic flow is asymptotically stable with respect to two-dimensional disturbances and is only neutrally stable with respect to three-dimensional disturbances. The instability of the periodic flow at Re$\approx$170 is due to a three-dimensional near-wake disturbance with a pronounced spanwise wavelength of $L=1.8D$. The origin of this three-dimensionality for very large and critical spanwise wavelengths is of fundamental importance for modeling purposes. It verifies that the three-dimensional local stability analysis for near-wake profiles[17] may be expected to describe three-dimensional wake structures—at least qualitatively. It also explains the large predictive power of the Ginzburg–Landau model for three-dimensional effects in the cylinder wake,[18,19] since this model can be considered as a continuous chain of dissipatively coupled near-wake oscillators.[20]

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