Low-dimensional modelling — POD Galërkin method

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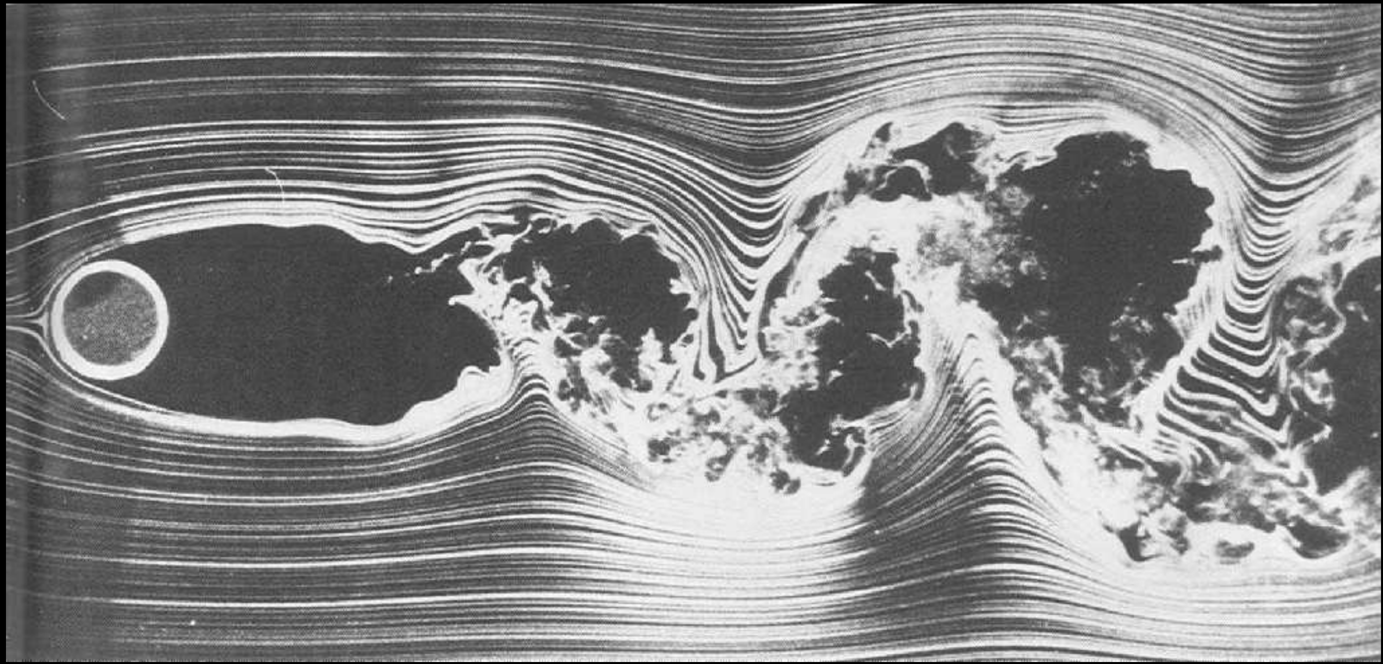

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CISM course "Reduced-Order Modelling for Flow Control", Udine, September 15–19, 2008

My lectures

1 (Mo) Motivation of Galerkin method, 2 examples

2 (Tu) Empirical Galerkin method based on POD

Purpose of this lecture

- Describe the standard POD Galerkin method
- Illustrate 'add-ons' for physical insights and control design

3 (Tu) POD-based Galerkin models of natural flow

4 (Th) POD-based Galerkin models of transient and actuated flow

5 (Th) Towards an attractor control

Overview

1. **Introduction**
2. **Proper Orthogonal Decomposition** $\mathbf{u} = \mathbf{u}_0 + \sum_{i=1}^{N} a_i \mathbf{u}_i$

   — Galerkin approximation for flow data
3. **Derivation of a dynamical system** $\dot{a}_i = f_i(a_1, \ldots, a_N)$

   — Galerkin projection on the Navier-Stokes equations
4. **Modal energy flow analysis**

   — 'Add-on' for physical insight
5. **Summary and outlook**

The art of Galerkin modeling

consists of making a good choice for its constitutive elements!

$$\mathbf{u}^{[0..N]} = \sum_{i=0}^{N} a_i \mathbf{u}_i, \qquad a_0 \equiv 1$$

$$\frac{d}{dt} a_i = \nu \sum_{j=0}^{N} l_{ij} a_j + \sum_{j,k=0}^{N} q_{ijk} a_j a_k$$

1. Basic mode $\mathbf{u}_0$:

Candidates:

- steady solution
- averaged solution
- mathematical flow

2. Hilbert space for $\mathbf{u}' = \mathbf{u} - \mathbf{u}_0$:

Candidates (solenoidal subsets):

- $\mathcal{L}^2(\Omega)$: $(\mathbf{u}, \mathbf{v})_\Omega = \int d\mathbf{x}\, \mathbf{u} \cdot \mathbf{v}$
- $\mathcal{L}^2_\sigma(\Omega)$: $(\mathbf{u}, \mathbf{v})_\Omega = \int d\mathbf{x}\, \sigma(\mathbf{x})\, \mathbf{u} \cdot \mathbf{v}$

3. Expansion modes $\mathbf{u}_i$:

modes	BC	NSE	Solution
mathematical	X		
physical	X	X	
empirical	**X**	**X**	**X**

4. Traditional Galerkin projection on NSE:

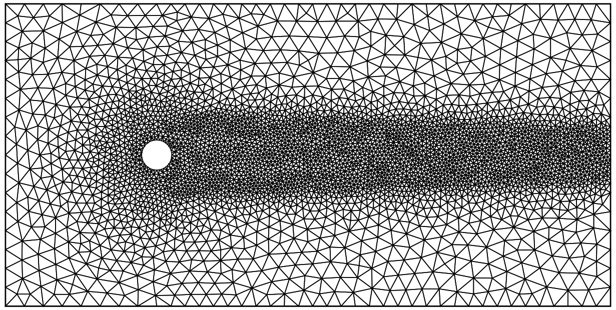
$\left(\mathbf{u}_i, \mathcal{N}(\mathbf{u}^{[0..N]})\right)_\Omega = 0 \qquad \mathcal{N}$: Navier-Stokes residual

Different Galerkin models

Computational Galerkin model

FEM by Morzyński

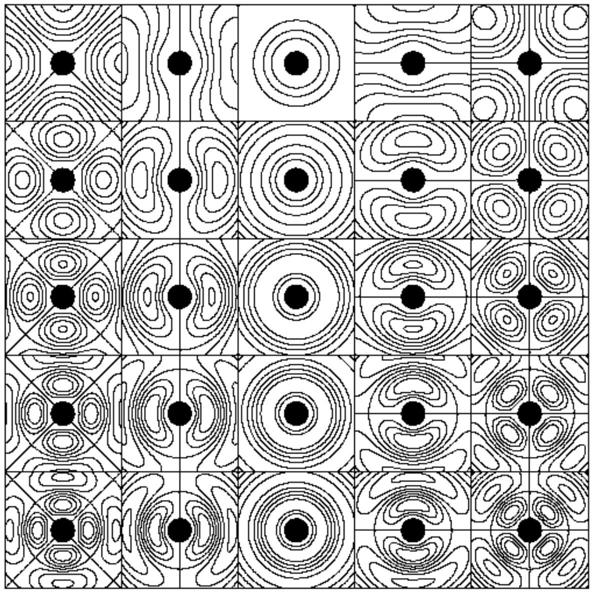
15838 grid points
31352 triangles



Mathematical Galerkin model

Noack & Eckelmann 1994 JFM

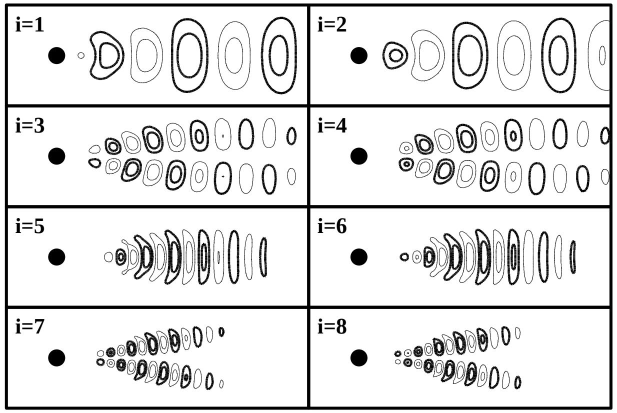
63 modes



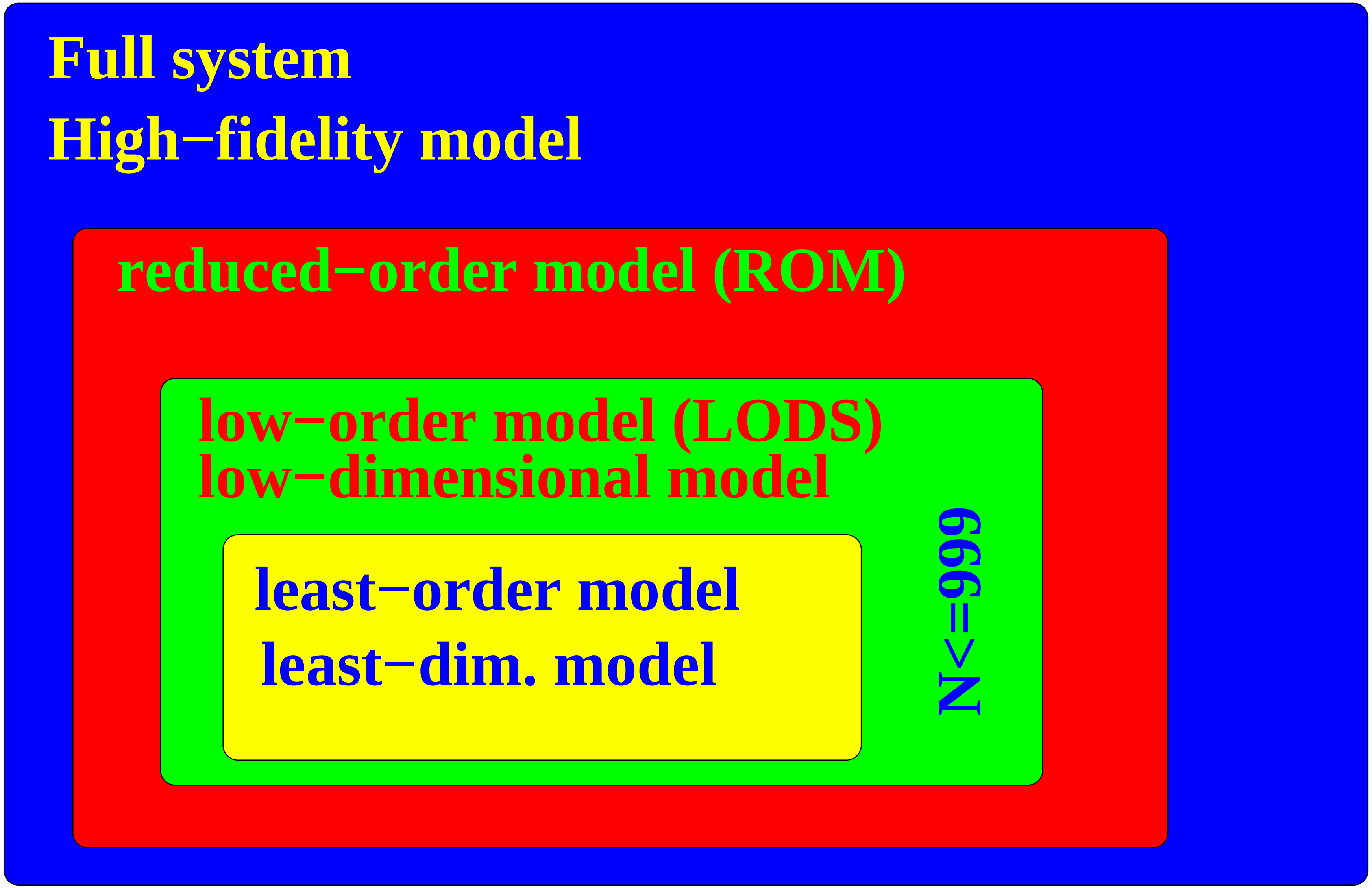
Empirical Galerkin model

Noack & friends 2003 JFM

6 modes



Reduced-, low- and least-order models



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    — Galerkin approximation for flow data

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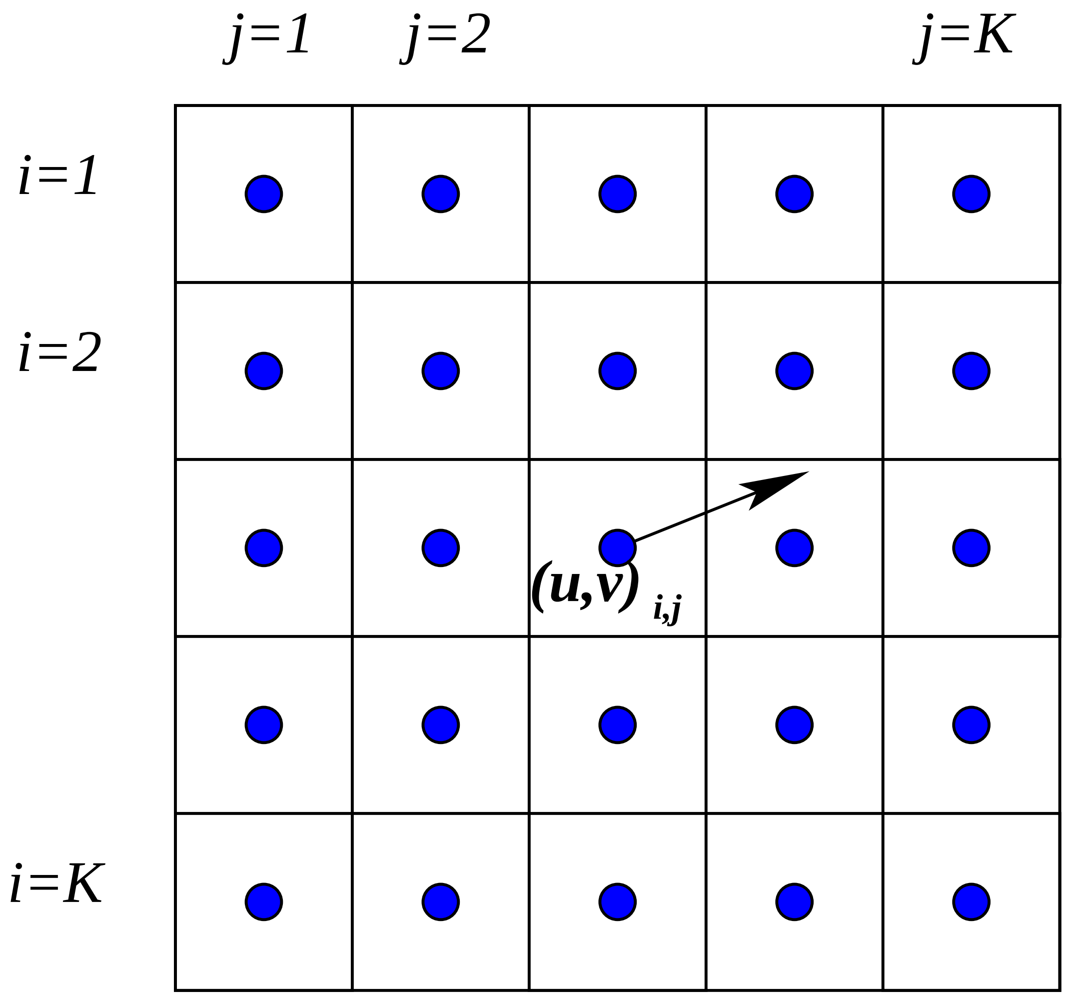
From flows to phase spaces

Discretized 2D flow



Phase space $\mathcal{R}^P$

$$
\begin{aligned}
u_1 &:= u_{11} \\
u_2 &:= u_{21} \\
u_3 &:= u_{31} \\
u_{K^2} &:= u_{KK} \\
u_{K^2+1} &:= v_{11} \\
\vdots & \quad \vdots \\
u_P &:= v_{KK} \\
P &= 2 \times K^2
\end{aligned}
$$

Autonomous SODE

Phase space $\mathcal{R}^P$

$$\mathbf{u} := (u_1, u_2, \ldots, u_P)^t$$

Dynamics

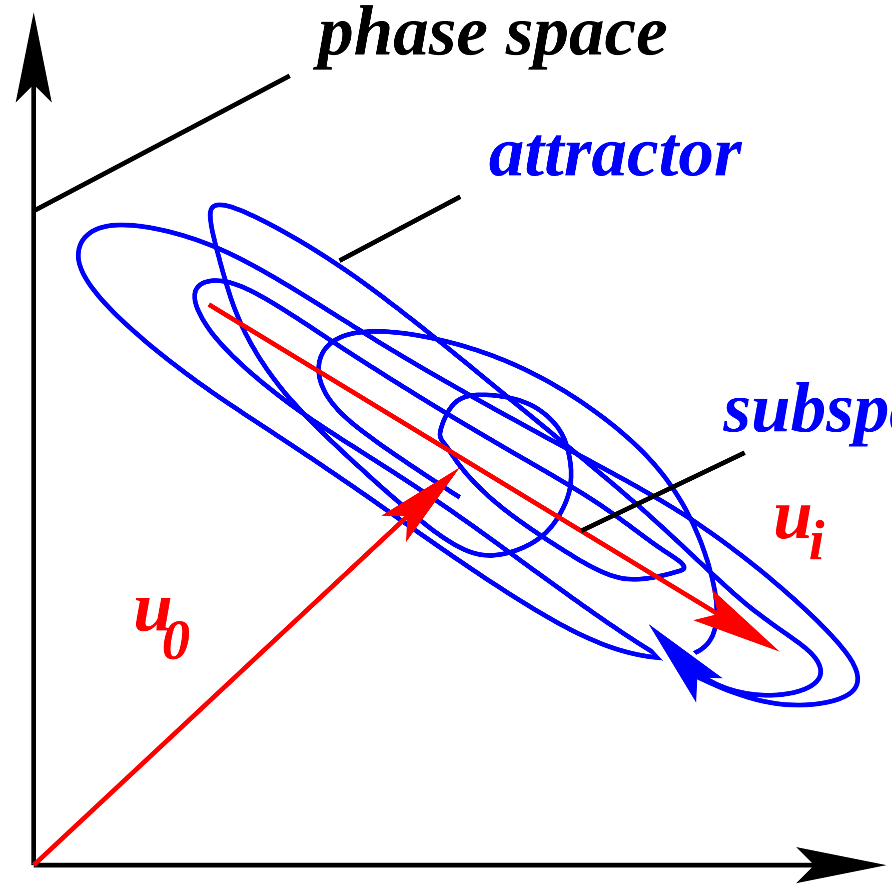
$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u})$$

Attractor $\mathcal{A} \subset \mathcal{R}^P$

$$\mathbf{u} \to \mathcal{A} \quad \text{as} \quad t \to \infty$$

characterized by ergodic measure $p_{\mathcal{A}}(\mathbf{u})$.

Idea of system reduction



Gaussian distribution of attractor

Attractor represented by trajectory $t \in [0, T] \mapsto \mathbf{u} \subset \mathcal{R}^P$

Gaussian probability distribution shall approximate attractor $p_{\mathcal{A}}(\mathbf{u})$

$$p_2(\mathbf{u}) = \frac{\sqrt{|Q|}}{(2\pi)^{P/2}} \exp\left[-\frac{1}{2}(\mathbf{u}-\mathbf{u}_0)^t \, Q \, (\mathbf{u}-\mathbf{u}_0)\right]$$

Matching the first statistical moments:

$$\mathbf{u}_0 = \overline{\mathbf{u}} = \int d\mathbf{u} \, p_2(\mathbf{u}) \, \mathbf{u} = \frac{1}{T}\int_0^T dt \, \mathbf{u}$$

Matching the 2nd statistical moments of the fluctuation: $\mathbf{u}' := \mathbf{u} - \mathbf{u}_0$

$$\boxed{Q = R^{-1}}$$

$$R := \overline{\mathbf{u}' \otimes \mathbf{u}'} = \int d\mathbf{u}' \, p_2(\mathbf{u}_0 + \mathbf{u}') \, \mathbf{u}' \otimes \mathbf{u}' = \begin{pmatrix} \overline{u_1' u_1'} & \overline{u_1' u_2'} & \ldots & \overline{u_1' u_P'} \\ \overline{u_2' u_1'} & \overline{u_2' u_2'} & \ldots & \overline{u_2' u_P'} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{u_P' u_1'} & \overline{u_P' u_2'} & \ldots & \overline{u_P' u_P'} \end{pmatrix}.$$

POD of attractor

Gaussian probability distribution of attractor $\mathcal{A} \subset \mathcal{R}^P$

$$p_2(\mathbf{u}) = \frac{\sqrt{|Q|}}{(2\pi)^{P/2}} \, e^{-\frac{1}{2}(\mathbf{u}-\mathbf{u}_0)^t \, Q \, (\mathbf{u}-\mathbf{u}_0)}, \quad \mathbf{u}_0 = \overline{\mathbf{u}}, \quad Q^{-1} = R = \overline{\mathbf{u}' \otimes \mathbf{u}'}$$

Principal axes

$$R\mathbf{u}_i = \lambda_i \mathbf{u}_i$$

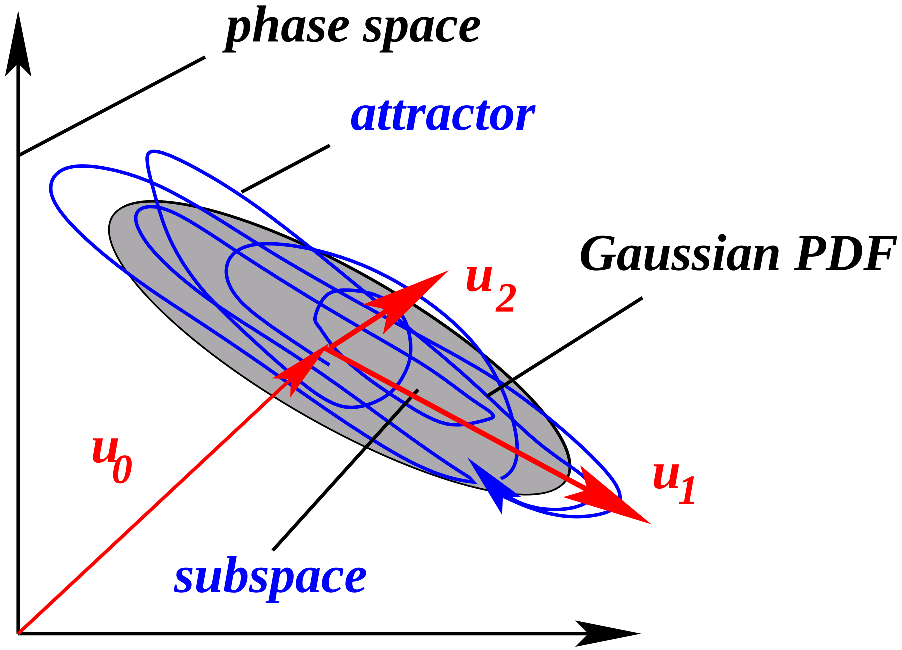
$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_P \geq 0.$

POD decomposition

$$\mathbf{u} = \mathbf{u}_0 + \sum_{i=1}^{P} a_i \mathbf{u}_i$$

Gaussian distribution

$$p_2(\mathbf{a}) = \frac{e^{-(\sum \frac{a_i^2}{\lambda_i})/2}}{\sqrt{(2\pi)^P \lambda_1 \ldots \lambda_P}}$$



$\mathbf{u}_i$: POD modes,

λ_i: POD eigenvalues

Properties of attractor POD

Trajectory

$$t \in [0,T] \mapsto \mathbf{u} \in \mathcal{R}^P$$

POD expansion $N \leq P$

$$\mathbf{u}^{[N]} = \mathbf{u}_0 + \sum_{i=1}^{N} a_i \mathbf{u}_i$$

Properties

- POD modes are orthogonal

$$\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$$

- Fourier coefficients

$$a_i = \mathbf{u}' \cdot \mathbf{u}_i$$

- Statistics of Fourier coefficients

$$\overline{a_i} = 0, \quad \overline{a_i a_j} = \delta_{ij} \lambda_i$$

Properties cont'd

- Correlation matrix

$$R := \overline{\mathbf{u}' \otimes \mathbf{u}'} = \sum_{i=1}^{P} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$$

- Fluctuation energy (trace of $R/2$)

$$K = \frac{1}{2} \overline{\mathbf{u}' \cdot \mathbf{u}'} = \frac{1}{2} \sum_{i=1}^{P} \lambda_i$$

- **Optimality property:** Let

$$\mathbf{v}^{[N]} = \mathbf{v}_0 + \sum_{i=1}^{N} b_i \mathbf{v}_i$$

be any other expansion then

$$\overline{\|\mathbf{u} - \mathbf{u}^{[N]}\|^2} \leq \overline{\|\mathbf{u} - \mathbf{v}^{[N]}\|^2}$$

POD analysis — Nomenclature

$\langle \mathbf{F} \rangle := \frac{1}{M} \sum_{m=1}^{M} \mathbf{F}^m$ . . . . ensemble average

$\langle \mathbf{F} \rangle_T := \frac{1}{T} \int_0^T dt\, \mathbf{F}$ . . . . time average

$(\mathbf{F})_\Omega := \int_\Omega dV\, \mathbf{F}$ . . . . volume integral

$[\mathbf{F}]_{\partial\Omega} := \int_\Omega d\mathbf{A} \cdot \mathbf{F}$ . . . . surface integral

POD in continuum flow limit

Limit $K \to \infty$ for $\mathcal{R}^P$ = 2D flow on equidistant $K \times K$ grid

Quantity	$\mathcal{R}^P$	**flow**
state space	$\mathbf{u} = (u_1, \ldots, u_P)$	$\mathbf{u}(\mathbf{x})$
inner prod.	$\mathbf{u} \cdot \mathbf{v} = \sum u_i v_i \Delta x^2$	$(\mathbf{u}, \mathbf{v})_\Omega = \int dV_\Omega \mathbf{u} \cdot \mathbf{v}$
correlation	$\mathbf{R} = \overline{\mathbf{u} \cdot \mathbf{u}}$	$\mathbf{R}(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{y}, t)}$
Fredholm equation	$\mathbf{R} \cdot \mathbf{u}_i = \lambda_i \mathbf{u}_i$	$\int d\mathbf{y}\, \mathbf{R}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}_i(\mathbf{y}) = \lambda_i\, \mathbf{u}_i(\mathbf{x})$
Exp.of $\mathbf{R}$	$\mathbf{R} = \sum \lambda_i \mathbf{u}_i \mathbf{u}_i$	$\mathbf{R}(\mathbf{x}, \mathbf{y}) = \sum \lambda_i \mathbf{u}_i(\mathbf{x})\, \mathbf{u}_i(\mathbf{y})$
Exp. of K	$K = \frac{1}{2} \overline{\mathbf{u} \cdot \mathbf{u}} = \frac{1}{2} \sum \lambda_i$	$K = \frac{1}{2} \overline{(\mathbf{u}, \mathbf{u})_\Omega} = \frac{1}{2} \sum \lambda_i$
GA	$\mathbf{u} = \mathbf{u}_0 + \sum a_i(t)\, \mathbf{u}_i$ $a_i = (\mathbf{u} - \mathbf{u}_0) \cdot \mathbf{u}_i$	$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}) + \sum a_i(t)\, \mathbf{u}_i(\mathbf{x})$ $a_i = (\mathbf{u} - \mathbf{u}_0, \mathbf{u}_i)_\Omega$

POD (spatial vs. temporal formulation)

	spatial POD	temporal POD
$f(x,t) =$	$= \sum_{i=1}^{N} a_i(t)\, u_i(x)$	$= \sum_{i=1}^{N} a_i^\star(t)\, u_i^\star(x)$
eigenmodes	$u_i(x)$	$a_i^\star(t)$
normalisation	$(u_i, u_j)_\Omega = \delta_{ij}$	$\langle a_i^\star\, a_j^\star \rangle_T = \delta_{ij}$
bi-orthog.	$\langle a_i\, a_j \rangle_T = \lambda_i\, \delta_{ij}$	$(u_i^\star, u_j^\star)_\Omega = \lambda_i\, \delta_{ij}$
coefficients	$a_i = (f, u_i)$	$u_i^\star = \langle f\, a_i^\star \rangle$
correlation tensor	$R(x,y)$ $= \langle f(x,t) f(y,t) \rangle_T$	$R^\star(t,s)$ $= (f(x,t), f(x,s))_\Omega$
Fredholm equation	$\int dy\, R(x,y)\, u_i(y)$ $= \lambda_i u_i(x)$	$\frac{1}{T} \int ds\, R(t,s)\, a_i^\star(s)$ $= \lambda_i a_i^\star(t)$
	Note: $a_i^\star = a_i / \sqrt{\lambda_i}$,	$u_i^\star = \sqrt{\lambda_i}\, u_i$
application	experimental data $P \ll M$	simulation data $P \gg M$

P: spatial dimension, M: number of snapshots

POD — mean flow

$$\mathbf{u}_0 = \frac{1}{M} \sum_{m=1}^{M} \mathbf{u}^m$$

Note that POD is a refined 2-points statistics up to second moments.

$$\langle \mathbf{u}'(\mathbf{x}, t) \otimes \mathbf{u}'(\mathbf{y}, t) \rangle = \sum_{i=1}^{\infty} \lambda_i \, \mathbf{u}_i(\mathbf{x}) \otimes \mathbf{u}_i(\mathbf{y})$$

Hence, a mimimum requirement to the snapshot ensemble is that the single points statistics of the first moments (mean values) and second centered moments (variances, $u_{\mathrm{rms}}(\mathbf{x})$,$v_{\mathrm{rms}}(\mathbf{x})$) are accurate.

POD — correlation matrix

$$C^{mn} = \frac{1}{M} (\mathbf{u}^m - \mathbf{u}_0, \mathbf{u}^n - \mathbf{u}_0)_\Omega$$

Note that the mean value is subtracted. Otherwise:

- The first POD mode is approximately the mean flow.
- The fluctuation has to be orthogonal to the mean flow.
- Convergence of the POD with $N \to \infty$ is not guaranteed.
- λ_i cannot be interpretated as variances.
- The POD Galerkin approximation $\mathbf{u} = \sum a_i \mathbf{u}_i$ does not fulfill the boundary conditions for arbitrary a_i.
- The POD Galerkin model is non physical, allows for varying oncoming velocities, etc.
- The beauty of POD modelling is lost.

POD — Eigenproblem

Fredholm equation (discretized in time domain):

$$\mathbf{C} \cdot \mathbf{e}_i = \lambda_i \, \mathbf{e}_i.$$

Here, $\mathbf{C} := (C^{mn}) \in \mathcal{R}^{M \times M}$, $\mathbf{e}_i := (e^i_1, e^i_2, \ldots, e^i_M)$.

- $\mathbf{C}$ is symmetric $\Rightarrow \lambda_i \in \mathcal{R}$ and

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

- $\mathbf{C}$ is positiv semi-definite $\Rightarrow \forall i\colon \lambda_i \geq 0$

POD — eigenmodes

$$\mathbf{u}_i := \frac{1}{\sqrt{M\,\lambda_i}} \sum_{m=1}^{M} e^i_m \, (\mathbf{u}^m - \mathbf{u}_0)$$

Validation:

- Check $\left(\mathbf{u}_i, \mathbf{u}_j\right)_\Omega = \delta_{ij}$
- Check $\mathcal{K} = \frac{1}{2} \left\langle \|\mathbf{u}'\|^2_\Omega \right\rangle = \frac{1}{2}\,\mathrm{trace}\, C = \frac{1}{2} \sum_{m=1}^{M} \lambda_m$

POD — Fourier coefficients

$$a_i(t_m) = a_i^m := \sqrt{\lambda_i \, M} \, e_m^i$$

Validation:

- Check $a_i^m = (\mathbf{u}^m - \mathbf{u}_0, \mathbf{u}_i)_\Omega$
- Check $\langle a_i \rangle = \frac{1}{M} \sum_{m=1}^{M} a_i^m = 0$
- Check $\langle a_i \, a_j \rangle = \frac{1}{M} \sum_{m=1}^{M} a_i^m a_j^m = \lambda_i \, \delta_{ij}$

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Galerkin projection on subspace

Evolution equation (EE) in $\mathcal{R}^P$

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u})$$

Galerkin approximation (GA): ONS, e.g. POD $N \leq P$

$$\mathbf{u}^{[0..N]} = \mathbf{u}_0 + \sum_{i=1}^{N} a_i \mathbf{u}_i$$

Goal=Galerkin system (GS)

$$\frac{da_i}{dt} = f_i(\mathbf{a}), \quad i = 1, \ldots, N$$

Galerkin projection (GP)

- Generally no exact derivation of GS possible.
- N equations need to be derived from the EE.

Traditional Galerkin projection (GP) of the time derivative:

$$\begin{aligned}
\mathbf{u}_i \cdot \frac{d\mathbf{u}^{[0..N]}}{dt} &= \mathbf{u}_i \cdot \left[\sum_{j=1}^{N} \frac{da_j}{dt} \mathbf{u}_j \right] \\
&= \sum_{j=1}^{N} \frac{da_j}{dt} \mathbf{u}_i \cdot \mathbf{u}_j \\
&= \frac{da_i}{dt}
\end{aligned}$$

exploiting $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$.

GP of the flow (r.h.s.)

$$\begin{aligned}
f_i(\mathbf{a}) &:= \mathbf{u}_i \cdot \mathbf{F}(\mathbf{u}^{[0..N]}) \\
&= \mathbf{u}_i \cdot \mathbf{F}\left(\mathbf{u}_0 + \sum_{j=1}^{N} a_j \mathbf{u}_j \right)
\end{aligned}$$

- **GS** is determined!

Weak form of Navier-Stokes equation

Navier-Stokes equation

$$R[\mathbf{u}] := \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \triangle \mathbf{u} + \nabla p = 0$$

Weak form of the Navier-Stokes equation. $\forall \mathbf{v}(\mathbf{x})$

$$(\mathbf{v}, R[\mathbf{u}]) := I(\mathbf{v}, \partial_t \mathbf{u}) + C(\mathbf{v}, \mathbf{u}, \mathbf{u}) - \nu\, D(\mathbf{v}, \mathbf{u}) + [\mathbf{v}, p]_{\partial\Omega} = 0$$

with bilinear and trilinear operators

$$\begin{aligned} I(\mathbf{u}, \mathbf{v}) &:= (\mathbf{u}, \mathbf{v})_{\Omega}\,, \\ D(\mathbf{u}, \mathbf{v}) &:= (\mathbf{u}, \triangle \mathbf{v})_{\Omega}\,, \\ C(\mathbf{u}, \mathbf{v}, \mathbf{w}) &:= (\mathbf{u}, \nabla \cdot (\mathbf{v} \otimes \mathbf{w}))_{\Omega} \\ [\mathbf{u}, f]_{\partial\Omega} &:= \oint_{\partial\Omega} d\mathbf{A} \cdot \mathbf{u}\, f \end{aligned}$$

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$I(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_\Omega = (\mathbf{u} \cdot \mathbf{v})_\Omega$ . . . . inner product

$D(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \triangle \mathbf{v})_\Omega$ . . . . for viscous term

$C(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \nabla \cdot [\mathbf{v} \otimes \mathbf{w}])_\Omega$ . . . . for convection term

Galerkin projection — local acceleration

Galerkin projection (GP) generates N equations for N unknown $a_i(t)$

$$\left(\mathbf{u}_i, R\left[\sum_{j=0}^{N} a_j(t)\,\mathbf{u}_j(\mathbf{x})\right]\right)_\Omega = 0, \quad i = 1,\ldots,N$$

GP of time derivative $\left(\mathbf{u}_i, \partial_t \sum_{j=0}^{N} a_j(t)\,\mathbf{u}_j(\mathbf{x})\right)_\Omega$

$$= \left(\mathbf{u}_i, \sum_{j=0}^{N} \frac{da_j}{dt}(t)\,\mathbf{u}_j(\mathbf{x})\right)_\Omega = \left(\mathbf{u}_i, \sum_{j=0}^{N} \frac{da_j}{dt}(t)\,\mathbf{u}_j(\mathbf{x})\right)_\Omega$$

$$= \sum_{j=1}^{N} \left(\mathbf{u}_i, \frac{da_j}{dt}\,\mathbf{u}_j(\mathbf{x})\right)_\Omega = \sum_{j=1}^{N} \frac{da_j}{dt}\,(\mathbf{u}_i, \mathbf{u}_j)_\Omega$$

$$= \sum_{j=1}^{N} \frac{da_j}{dt}\,\delta_{ij} = \frac{da_i}{dt}$$

Galerkin projection — viscous term

GP of viscous term $\nu \left(\mathbf{u}_i, \triangle \sum_{j=0}^{N} a_j(t)\, \mathbf{u}_j(\mathbf{x}) \right)_\Omega$

$$= \nu\, D \left(\mathbf{u}_i, \sum_{j=0}^{N} a_j\, \mathbf{u}_j \right)$$

$$= \nu \sum_{j=0}^{N} D \left(\mathbf{u}_i, a_j\, \mathbf{u}_j \right)$$

$$= \nu \sum_{j=0}^{N} a_j\, \underbrace{D \left(\mathbf{u}_i, \mathbf{u}_j \right)_\Omega}_{=: l_{ij}}$$

$$= \nu \sum_{j=0}^{N} l_{ij} a_j$$

Galerkin projection — convection term

GP of convection term $-\left(\mathbf{u}_i, \nabla \cdot \left[\sum_{j=0}^{N} a_j\, \mathbf{u}_j \otimes \sum_{k=0}^{N} a_k\, \mathbf{u}_k\right]\right)_\Omega$

$$= -C\left(\mathbf{u}_i, \sum_{j=0}^{N} a_j\, \mathbf{u}_j, \sum_{k=0}^{N} a_k\, \mathbf{u}_k\right)$$

$$= -\sum_{j=0}^{N}\sum_{k=0}^{N} C\left(\mathbf{u}_i, a_j\, \mathbf{u}_j, a_k\, \mathbf{u}_k,\right)$$

$$= -\sum_{j=0}^{N}\sum_{k=0}^{N} a_j\, a_k\, \underbrace{C\left(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\right)}_{=:q_{ijk}^c}$$

$$= \sum_{j,k=0}^{N} q_{ijk}^c\, a_j\, a_k$$

Galerkin projection — pressure term

Pressure-Poisson equation

$$\triangle p = -\nabla \cdot \nabla \cdot \mathbf{u} \otimes \mathbf{u}$$

Pressure expansion *(Noack et al. 2005 JFM)*

$$p^{[0..N]}(\mathbf{x},t) = \sum_{j=0}^{N} \sum_{j=0}^{N} p_{jk}(\mathbf{x})\, a_j(t)\, a_k(t)$$

GP of pressure term $-\left(\mathbf{u}_i, \nabla p\right)_\Omega = -\left[\mathbf{u}_i, p^{[0..N]}\right]_{\partial\Omega}$

$$\begin{aligned} &= -\left[\mathbf{u}_i, \sum_{j=0}^{N} \sum_{k=0}^{N} p_{jk} a_j a_k\right]_{\partial\Omega} \\ &= -\sum_{j=0}^{N} \sum_{k=0}^{N} a_j\, a_k \underbrace{\left[\mathbf{u}_i, p_{jk}\right]_{\partial\Omega}}_{=:-q^p_{ijk}} = \sum_{j,k=0}^{N} q^p_{ijk}\, a_j\, a_k \end{aligned}$$

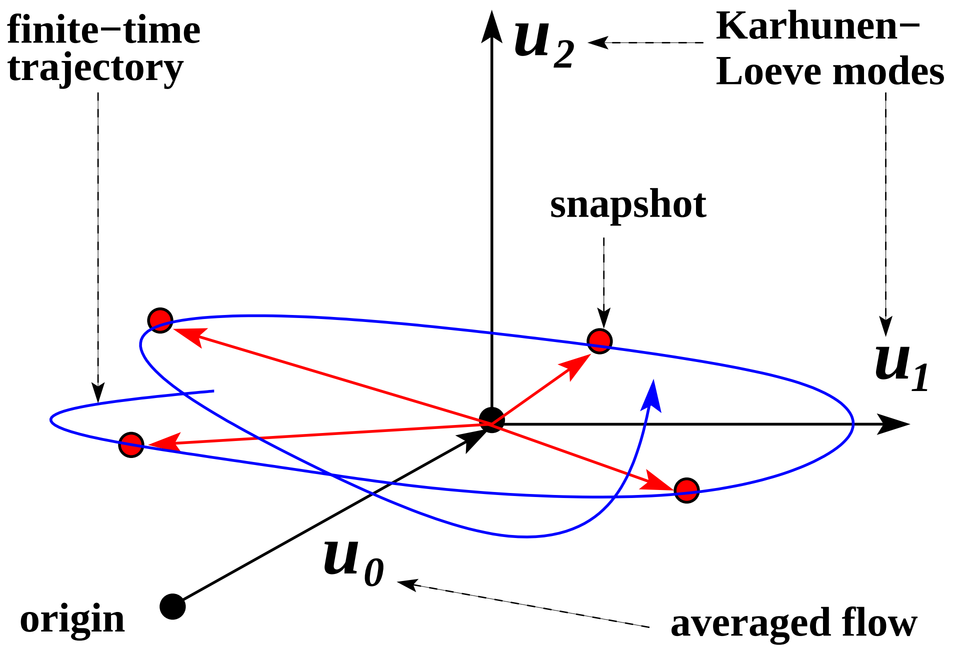
Galerkin method — summary

Galerkin method

$$\begin{array}{ccccccc} \mathbf{u} & \rightarrow & \partial_t \mathbf{u} & = \nu \triangle \mathbf{u} & -\nabla(\mathbf{u}\mathbf{u}) & -\nabla p \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{u} = \sum_{i=0}^{N} a_i \mathbf{u}_i & \rightarrow & \frac{da_i}{dt} & = \nu \sum_{j=0}^{N} l_{ij} a_j & + \sum_{j,k=0}^{N} (q_{ijk} & + q_{ijk}^{\pi})\, a_j\, a_k \end{array}$$

Galerkin approximation

(Karhunen-Loève decomposition, principal axes)



Galerkin projection

$$(\mathbf{u}, \mathbf{v})_\Omega := \int dV\, \mathbf{u} \cdot \mathbf{v}$$

$$\begin{aligned} (\mathbf{u}_i, \partial_t \mathbf{u})_\Omega &= \int dV\ \mathbf{u}_i \cdot \partial_t \left(\sum_{j=0}^{N} a_j \mathbf{u}_j \right) \\ &= \sum_{j=1}^{N} \frac{da_i}{dt} \int dV\ \mathbf{u}_i \cdot \mathbf{u}_j \\ &= \frac{d}{dt} a_i \end{aligned}$$

Overview

1. **Introduction**
2. **Proper Orthogonal Decomposition** $\mathbf{u} = \mathbf{u}_0 + \sum_{i=1}^{N} a_i \mathbf{u}_i$

   — Galerkin approximation for flow data
3. **Derivation of a dynamical system** $\dot{a}_i = f_i(a_1, \ldots, a_N)$

   — Galerkin projection on the Navier-Stokes equations
4. **Modal energy flow analysis**

   — 'Add-on' for physical insight
5. **Summary and outlook**

Global energy flow analysis

— ≡ *Noack, Papas & Monkewitz (2005) JFM* —

In a nutshell:

Reynolds decomposition .... $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}', \quad \mathbf{u}_0 = \overline{\mathbf{u}}$

Navier-Stokes equation (NSE) .... $\mathcal{R}[\mathbf{u}] = 0$

Reynolds equation (RE) .... $\overline{\mathcal{R}[\mathbf{u}_0 + \mathbf{u}']} = 0$

weak formulation of NSE .... $\forall \mathbf{v}: \ (\mathbf{v}, \mathcal{R}[\mathbf{u}])_\Omega = 0$

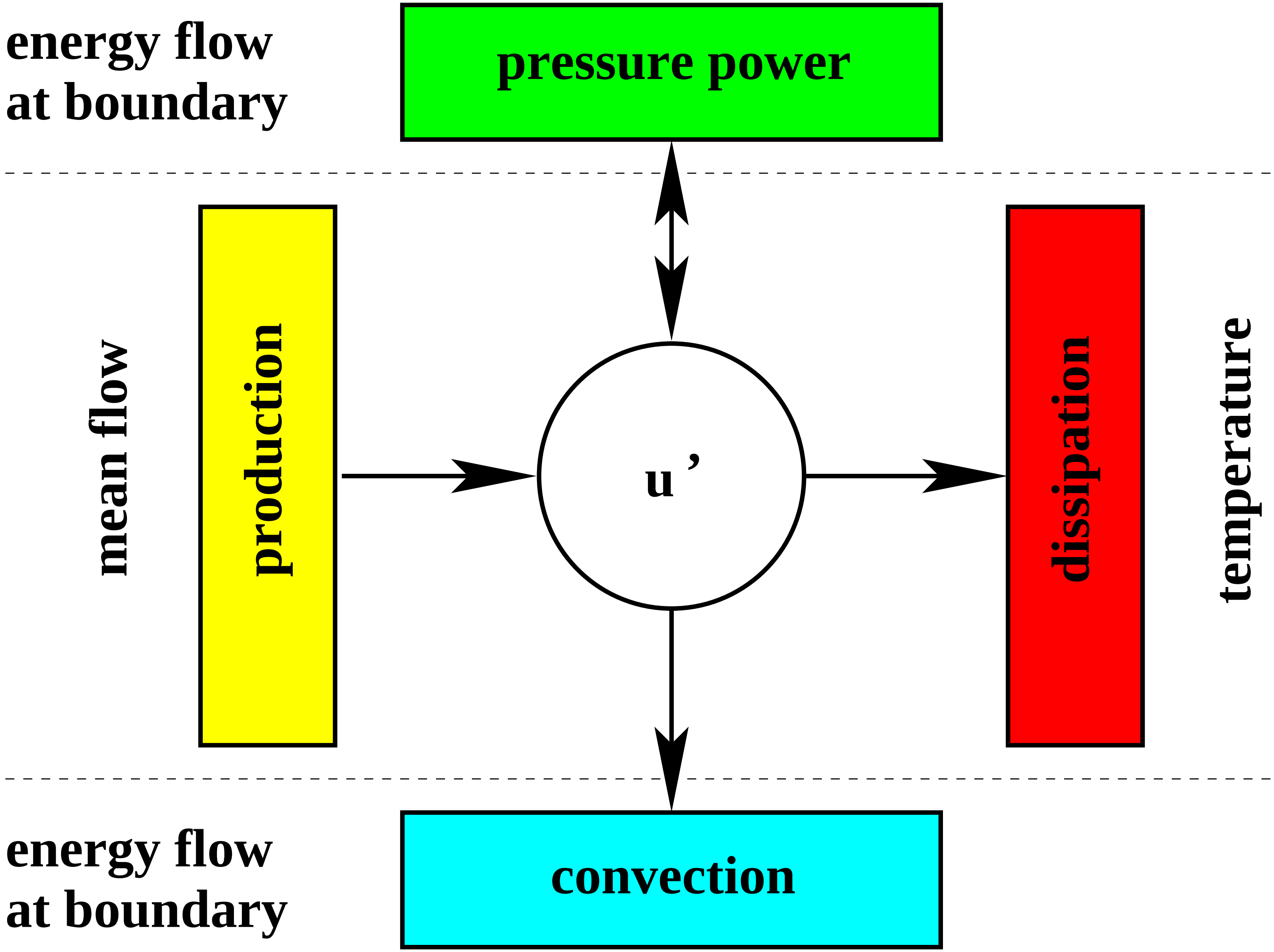
Balance eq. of turbulent kinetic energy (TKE): $\overline{(\mathbf{u}', \mathcal{R}[\mathbf{u}_0 + \mathbf{u}'])_\Omega} = 0$

In some detail:

NSE	NSE II	RE	NSE II-RE	TKE	
$\partial_t \mathbf{u} =$	$\partial_t \mathbf{u}' =$	$0 =$	$\partial_t \mathbf{u}' =$	$\frac{d}{dt} \int dV \frac{1}{2} \overline{\Vert\mathbf{u}'\Vert^2} =$	$dK/dt =$
$-\nabla \cdot \mathbf{u}\mathbf{u}$	$-\nabla \cdot \mathbf{u}_0\mathbf{u}_0$ $-\nabla \cdot \mathbf{u}'\mathbf{u}_0$ $-\nabla \cdot \mathbf{u}_0\mathbf{u}'$ $-\nabla \cdot \mathbf{u}'\mathbf{u}'$	$-\nabla \cdot \mathbf{u}_0\mathbf{u}_0$ $-\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}$	 $-\nabla \cdot \mathbf{u}'\mathbf{u}_0$ $-\nabla \cdot \mathbf{u}_0\mathbf{u}'$ $-\nabla \cdot \mathbf{u}'\mathbf{u}'$ $+\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}$	 $-\int dV \overline{\mathbf{u}'\mathbf{u}'} : \nabla \mathbf{u}_0$ $- \oint d\mathbf{A} \cdot \mathbf{u}_0 \frac{1}{2} \overline{\Vert\mathbf{u}'\Vert^2}$ $-\int dV \overline{\mathbf{u}' \cdot \nabla \cdot \mathbf{u}'\mathbf{u}'}$	 $+\mathcal{P}$ $+\mathcal{C}$ $+\mathcal{T}$
$+\nu \triangle \mathbf{u}$	$+\nu \triangle \mathbf{u}_0$ $+\nu \triangle \mathbf{u}'$	$+\nu \triangle \mathbf{u}_0$	 $+\nu \triangle \mathbf{u}'$	 $+\nu \int dV \overline{\mathbf{u}' \cdot \triangle \mathbf{u}'}$	 $+\mathcal{D}$
$-\nabla p$	$-\nabla p_0$ $-\nabla p'$	$-\nabla p_0$	 $-\nabla p'$	 $- \oint d\mathbf{A} \cdot \overline{\mathbf{u}' p'}$	 $+\mathcal{F}$

Global energy flow analysis

— Noack, Papas & Monkewitz (2005) JFM —



Modal fluid dynamics

— Noack, Papas & Monkewitz (2005) JFM —

In a nutshell:

Galerkin approximation . $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}', \quad \mathbf{u}_0 := \overline{\mathbf{u}}, \quad \mathbf{u}' := \sum_{i=1}^{N} a_i \mathbf{u}_i$

Navier-Stokes Eq. . . . . $\mathcal{R}(\mathbf{u}) = 0$

Galerkin system . . . . $\left(\mathbf{u}_i, \mathcal{R}(\mathbf{u}^{[N]})\right)_\Omega = 0$

Modal energy flow balance $\overline{\left(a_i\mathbf{u}_i, \mathcal{R}(\mathbf{u}^{[N]})\right)_\Omega} = 0$

Global energy flow balance $\overline{\left(\mathbf{u}', \mathcal{R}(\mathbf{u}^{[N]})\right)_\Omega} = 0$

$$\overline{F} = \frac{1}{T}\int_0^T dt\, F$$

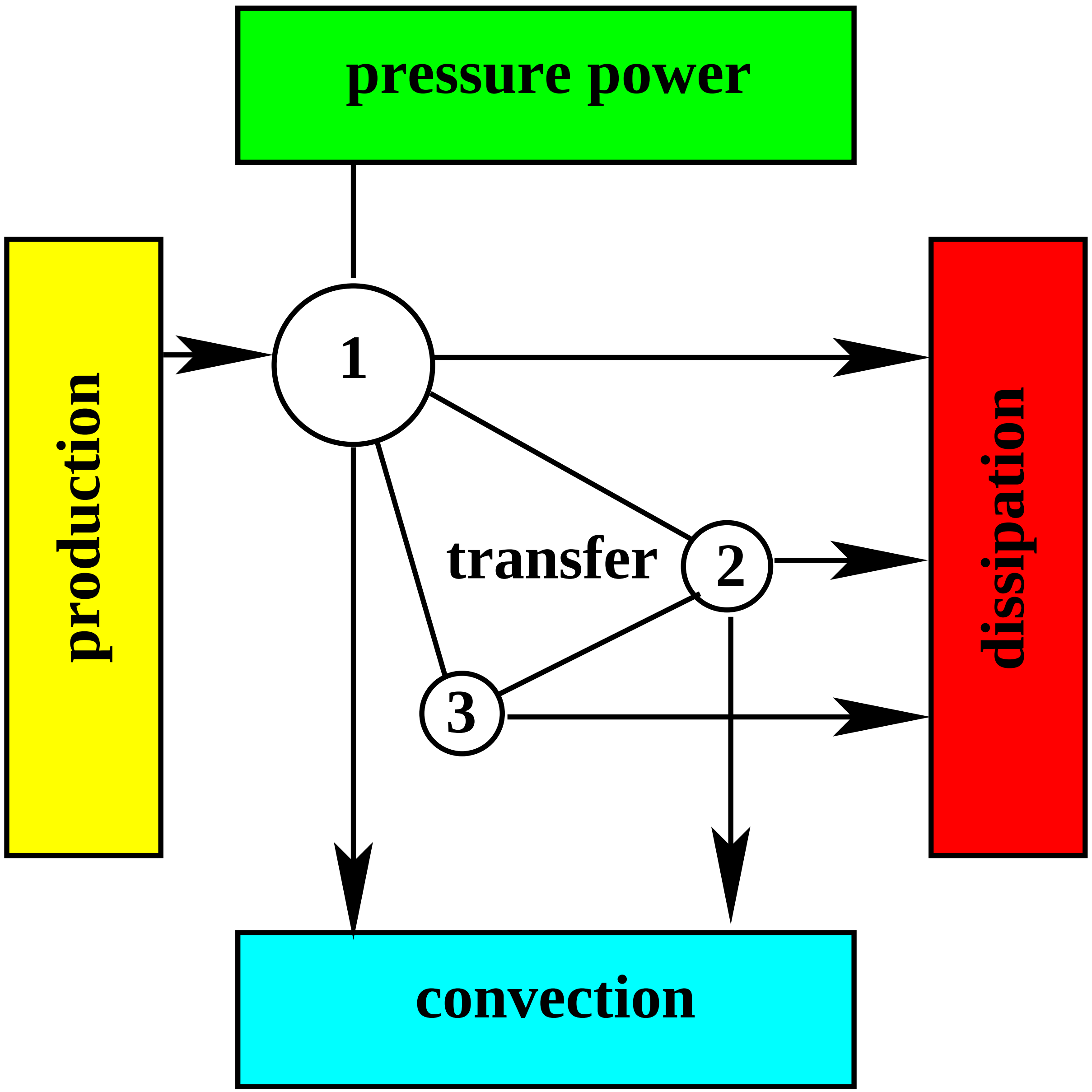
$$(\mathbf{u}, \mathbf{v})_\Omega := \int_\Omega dV\, \mathbf{u}\cdot\mathbf{v}$$

Im some detail:

NSE	NSE II	GS	modal E	
$\partial_t \mathbf{u} =$	$\partial_t \mathbf{u}' =$	$da_i/dt =$	$\frac{d}{dt}\overline{a_i^2}/2 =$	$d\, K_i /dt =$
$-\nabla\cdot\mathbf{u}\mathbf{u}$	$-\nabla\cdot\mathbf{u}_0\mathbf{u}_0$	$+q_{i00}$		
	$-\nabla\cdot\mathbf{u}'\mathbf{u}_0$	$+\sum_{j=1}^N q_{ij0}a_j$	$+2q_{ii0}\, K_i$	$+\mathcal{P}_i$
	$-\nabla\cdot\mathbf{u}_0\mathbf{u}'$	$+\sum_{j=1}^N q_{i0j}a_j$	$+2q_{i0i}\, K_i$	$+\mathcal{C}_i$
	$-\nabla\cdot\mathbf{u}'\mathbf{u}'$	$+\sum_{j,k=1}^N q_{ijk}a_j a_k$	$+\sum_{j,k=1}^N q_{ijk}\overline{a_i a_j a_k}$	$+\mathcal{T}_i$
$+\nu\triangle\mathbf{u}$	$+\nu\triangle\mathbf{u}_0$	$+\nu l_{i0}$		
	$+\nu\triangle\mathbf{u}'$	$+\nu\sum_{j=1}^N l_{ij}a_j$	$+2\nu l_{ii}\, K_i$	$+\mathcal{D}_i$
$-\nabla p$	$-\nabla p$	$+\sum_{j,k=1}^N q^\pi_{ijk}a_j a_k$	$+\sum_{j,k=1}^N q^\pi_{ijk}\overline{a_i a_j a_k}$	$+\mathcal{F}_i$

Modal enery flow analysis

— *Noack, Papas & Monkewitz (2005) JFM* —



$$
\begin{array}{ccc}
P & = & \sum P_i \\
+ & & + \\
D & = & \sum D_i \\
+ & & + \\
C & = & \sum C_i \\
+ & & + \\
T & = & \sum T_i \\
+ & & + \\
F & = & \sum F_i \\
= & & = \\
0 & = & 0
\end{array}
$$

Modal energy flow balance

$$\frac{dK_i}{dt} = P_i + D_i + C_i + T_i + F_i = 0$$

$$\begin{aligned}
\frac{dK_i}{dt} &= \frac{1}{2}\frac{d}{dt}a_i^2 = \left\langle \left(\mathbf{u}^{[i]}, \partial_t \mathbf{u}^{[0..N]}\right)_\Omega \right\rangle = 0 \\
D_i &= \nu\, l_{ii}\, \lambda_i = +\nu \left\langle D\left(\mathbf{u}^{[i]}, \mathbf{u}^{[0..N]}\right)\right\rangle \\
P_i &= q_{ii0}^c \lambda_i = -\left\langle C\left(\mathbf{u}^{[i]}, \mathbf{u}^{[0]}, \mathbf{u}^{[0..N]}\right)\right\rangle \\
C_i &= q_{i0i}^c \lambda_j = -\left\langle C\left(\mathbf{u}^{[i]}, \mathbf{u}^{[0..N]}, \mathbf{u}^{[0]}\right)\right\rangle \\
T_i &= \sum_{j=1}^{N}\sum_{k=1}^{N} q_{ijk}^c \langle a_i\, a_j\, a_k\rangle = -\left\langle C\left(\mathbf{u}^{[i]}, \mathbf{u}^{[1..N]}, \mathbf{u}^{[1..N]}\right)\right\rangle \\
F_i &= \sum_{j=0}^{N}\sum_{k=0}^{N} q_{ijk}^p \langle a_i\, a_j\, a_k\rangle = -\left\langle \left[\mathbf{u}^{[i]}, p^{[0..N]}\right]_{\partial\Omega}\right\rangle
\end{aligned}$$

Modal energy flow balance

Galerkin projection (GP) onto $\mathbf{u}^{[i]} = a_i \mathbf{u}_i$ generates N equations for N modal energy flow budgets

$$\left(\mathbf{u}^{[i]}, R\left[\sum_{j=0}^{N} a_j(t)\, \mathbf{u}_j(\mathbf{x}) \right] \right)_{\Omega} = 0, \quad i = 1, \ldots, N$$

$\Rightarrow$

$$\begin{aligned} \left(\mathbf{u}^{[i]}, \partial_t \mathbf{u}^{[0..N]} \right)_{\Omega} + C\left(\mathbf{u}^{[i]}, \mathbf{u}^{[0..N]}, \mathbf{u}^{[0..N]} \right) & \\ -\nu\, D\left(\mathbf{u}^{[i]}, \mathbf{u}^{[0..N]} \right) + \left[\mathbf{u}^{[i]}, p^{[0..N]} \right]_{\partial\Omega} &= 0 \end{aligned}$$

Here, the modal energy flow is monitored instantaneously.

POD dynamical system
— modal balance equations

Galerkin system

$$\frac{da_i}{dt} = \nu \sum_{j=0}^{N} l_{ij}\, a_j + \sum_{j=0}^{N} \sum_{k=0}^{N} q_{ijk}\, a_j\, a_k \tag{1}$$

Galerkin-Reynolds equation = $\langle(1)\rangle$

using $\langle a_i \rangle = 0$ and $\langle a_i\, a_j \rangle = \lambda_i \delta_{ij}$

$$0 = \nu l_{i0} + \sum_{j=0}^{N} q_{ijj} \lambda_j \tag{2}$$

Modal energy flow equation = $\langle a_i \times (1) \rangle$

$$0 = \nu l_{ii} \lambda_i + \sum_{j=0}^{N} \sum_{k=0}^{N} q_{ijk}\, \langle a_i\, a_j\, a_k \rangle \tag{3}$$

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POD Galerkin method — Summary 1

(-1) Given M snapshots $\{\mathbf{u}^m = \mathbf{u}(\mathbf{x}, t_m)\}_{m=1}^{M}$

(0) Write subroutines for

$I(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v})_\Omega\,,$	$D(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \triangle \mathbf{v})_\Omega\,,$
$C(\mathbf{u}, \mathbf{v}, \mathbf{w}) := (\mathbf{u}, \nabla \cdot (\mathbf{v} \otimes \mathbf{w}))_\Omega$	$[\mathbf{u}, f]_{\partial\Omega} := \oint_{\partial\Omega} d\mathbf{A} \cdot \mathbf{u}\, f$

(1) Compute mean flow $\mathbf{u}_0 = \frac{1}{M} \sum_{m=1}^{M} \mathbf{u}^m$

(2) Compute correlation matrix $\mathbf{C}^{mn} = I\left(\mathbf{u}^m - \mathbf{u}_0, \mathbf{u}^n - \mathbf{u}_0\right)$

(3) Perform spectral analysis $\mathbf{C}\, \mathbf{e}_i = \lambda_i\, \mathbf{e}_i$, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$

(4) Compute POD modes $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i M}} \sum_{m=1}^{M} e^i_m \left(\mathbf{u}^m - \mathbf{u}_0\right)$

(5) Compute Fourier coefficients $a_i^m := \sqrt{\lambda_i\, M}\, e^i_m$

POD Galerkin method — Summary 2

(6) Perform Galerkin projection

$$\dot{a}_i = \nu \sum_{j=0}^{N} l_{ij}\, a_j + \sum_{j,k=0}^{N} q^c_{ijk}\, a_i\, a_j$$

where $l_{ij} := D(\mathbf{u}_i, \mathbf{u}_j)$ and $q^c_{ijk} := C(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k)$.

(7) Compute modal energy flow analysis

$$dK_i/dt = 0 = P_i + D_i + C_i + T_i + F_i,$$

where $K_i = \overline{a_i^2}/2$ $P_i = q_{ii0}\, \lambda_i$, $C_i = q_{i0i}\, \lambda_i$, $D_i = \nu\, l_{ii}\, \lambda_i$,
$T_i = q^c_{ijk}\, \overline{a_i\, a_j\, a_k}$, $F_i \approx 0$ (often).

POD model for flow control

- **Galerkin approximation** $\mathbf{u} = \sum_{i=1}^{N} a_i \mathbf{u}_i$, $\mathbf{a} = (a_1, a_2, \ldots, a_N)$

  . . . . data compression

  $O(1)$ Million grid points $\Rightarrow$ $O(10)$ Fourier coefficients
- **Modal energy flow analysis** $\frac{dE_i}{dt} = P_i + D_i + C_i + T_i + F_i$

  . . . . understanding of mode sociology

  . . . . Quasi-spectral understanding of data
- **Galerkin system** $\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a})$ . . . . efficient time integration

  Computation time drastically reduced
- **Dynamics exploration** $\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a})$ . . . . data base extrapolation
- **Observer design**: $S(t) = \sum a_i(t)\, \mathbf{u}_i(\mathbf{x}) \cdot \mathbf{e}_x \Rightarrow \hat{\mathbf{a}}(t) \Rightarrow \mathbf{u}(\mathbf{x}, t)$

  $\frac{d\hat{\mathbf{a}}}{dt} = \mathbf{f}(\hat{\mathbf{a}}) + L\,(S - \widehat{S})$ . . . . completion of experimental data
- **Control design**: $\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}, G)$ . . . . MIMO control law G=G(S)

  Linearization rarely works!

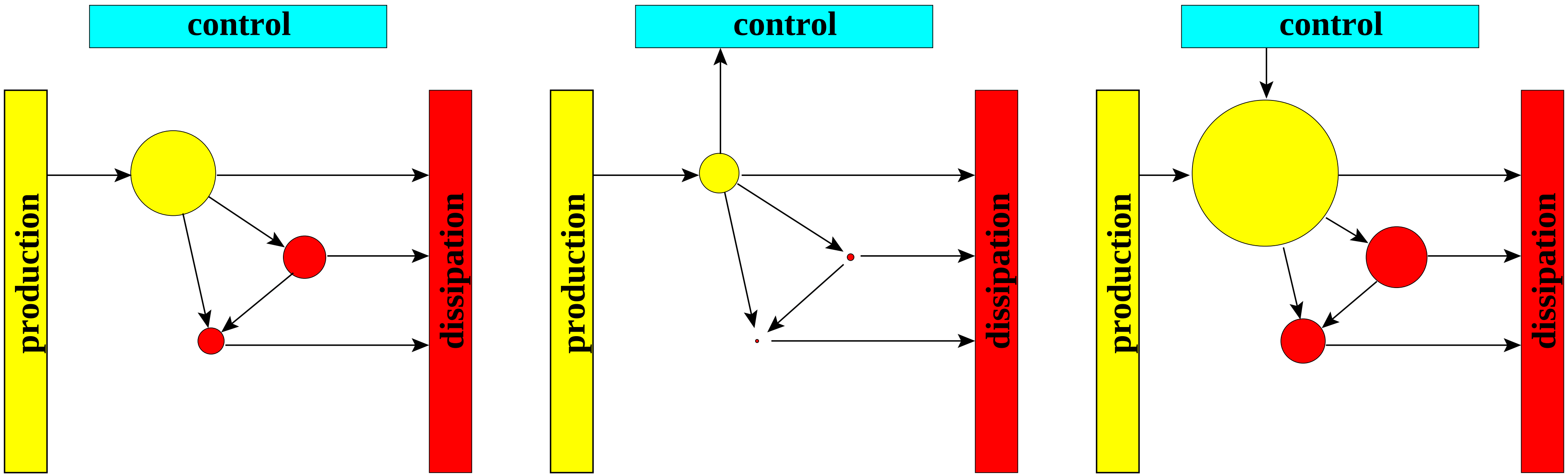
  Only a simple tune-able control law *structure*

  is generally applicable to experiment.
- **Real world check**: Only simple (low-dimensional) control strategies will survive real-world online-capability and robustness

Feedback flow control strategies based on modal energy flow analysis

— single active mode —

a) natural flow b) suppression c) enhancement

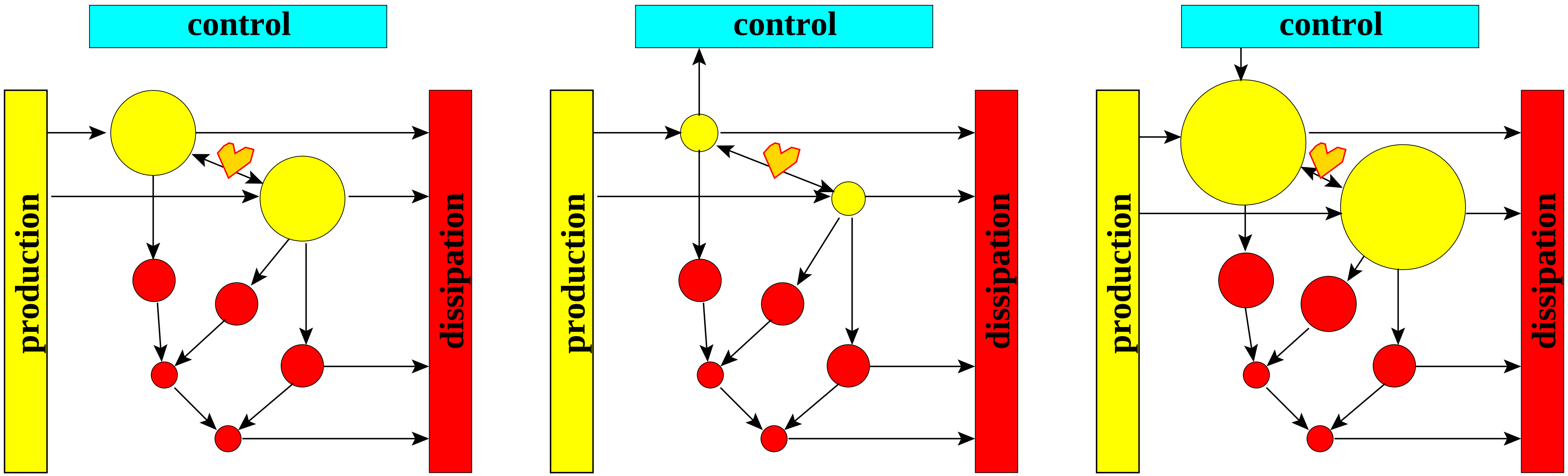


Example: von Kármán vortex shedding

Feedback flow control strategies based on modal energy flow analysis

— 2 synergizing active modes —

a) natural flow b) suppression c) enhancement

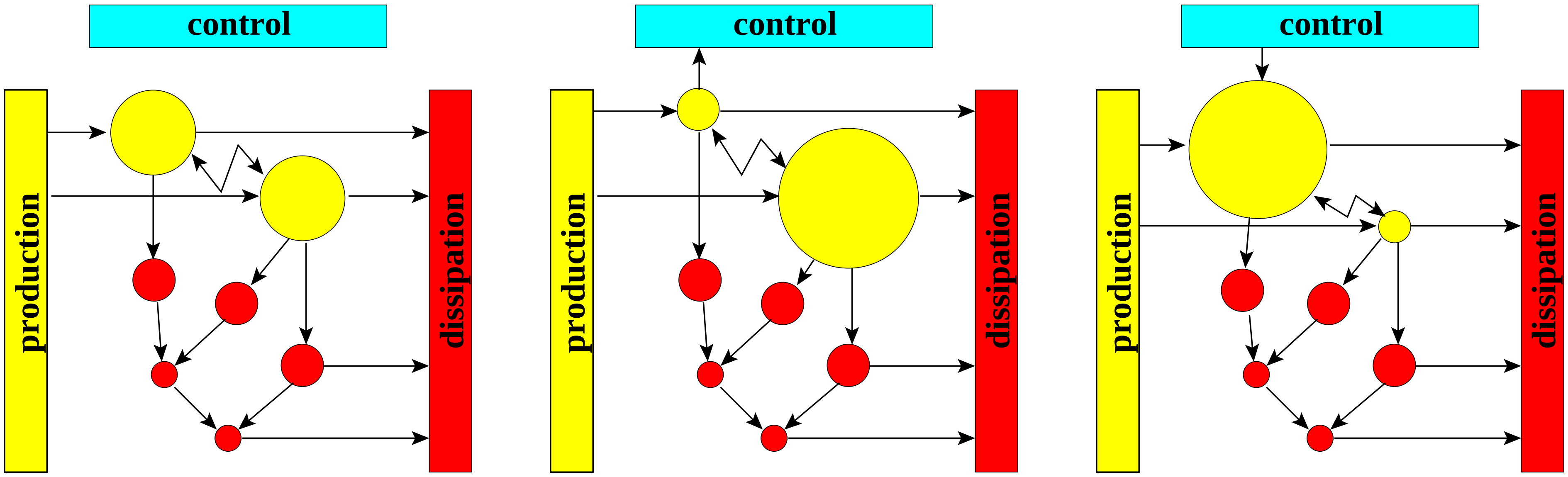


Example: vortex pairing (mode 1) of Kelvin-Helmholtz instability (mode 2)

Feedback flow control strategies based on modal energy flow analysis

— 2 competing active modes —

a) natural flow   b) suppression   c) enhancement



Example: Shear-layer vortices (mode 1) excited by cylinder rotation suppress vortex shedding (mode 2)

[Bergmann & Cordier 2005 PF]