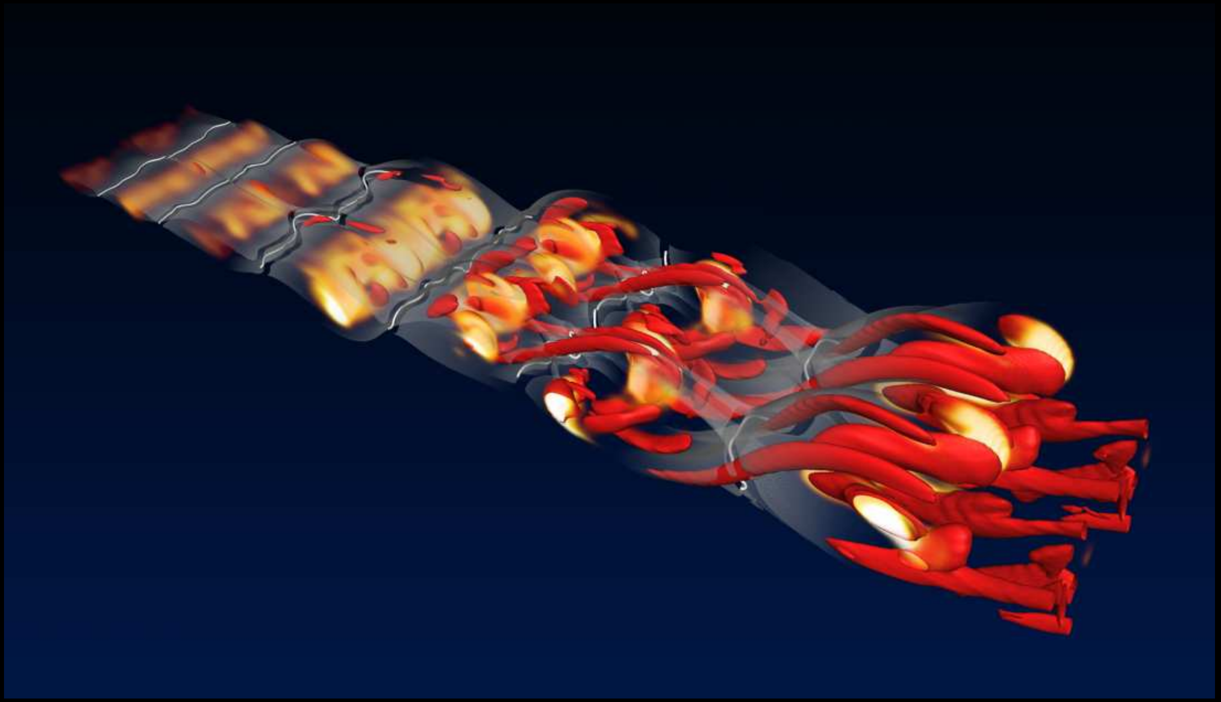
Low-dimensional modelling
— Post-transient natural flow



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My lectures

1 (Mo) Motivation of Galerkin method, 2 examples

2 (Tu) Empirical Galerkin method based on POD

3 (Tu) POD-based Galerkin models of natural flow

Purpose of this lecture:

- Show POD Galerkin models for natural flows
- Auxiliary models in the dynamical system: pressure, turbulence, ...

4 (Th) POD-based Galerkin models of transient and actuated flow

5 (Th) Towards an attractor control

Overview

1. Cylinder wake (standard method)

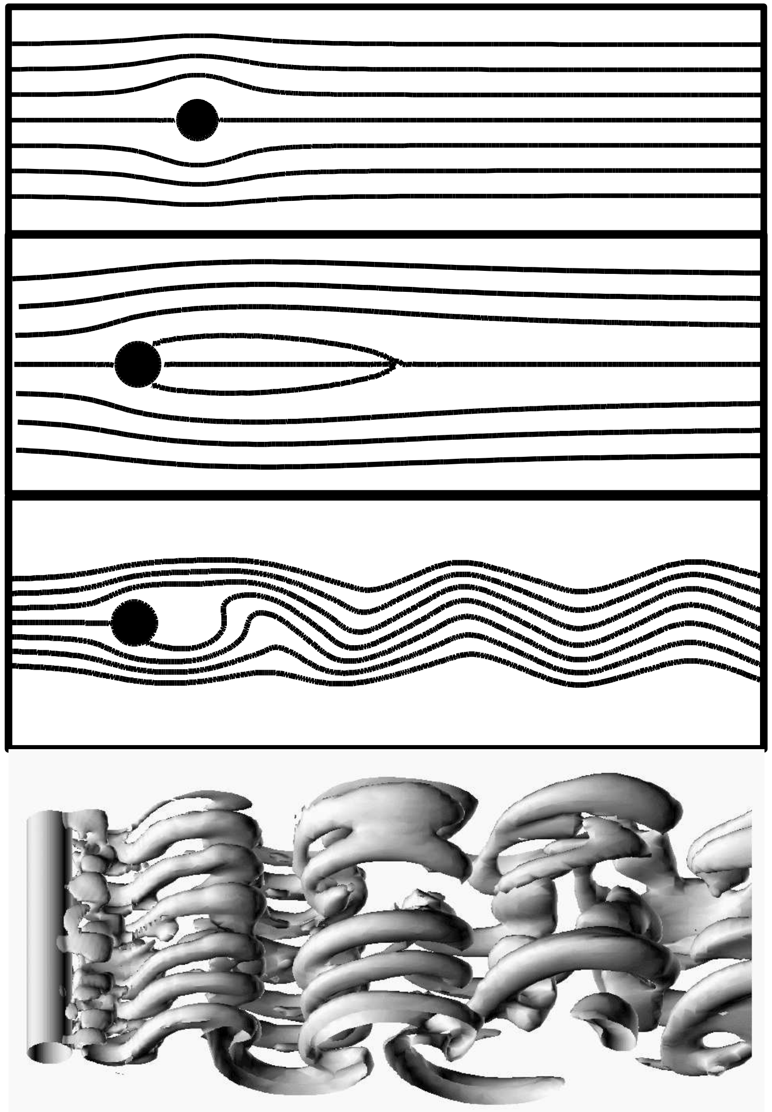
2. Laminar shear layer (+pressure model)

3. Turbulent mixing layer (+turbulence model)

4. Take home messages

Phenomenogram of cylinder wake

Reynolds number $Re = \frac{UD}{\nu}$



$Re < 4$ — 2D steady flow without vortex pair

$Re < 47$ — 2D steady flow *with* vortex pair

$Re < 180$ — 2D vortex shedding

$180 < Re$ — 2D vortex shedding superimposed by 3D modes / fluctuations

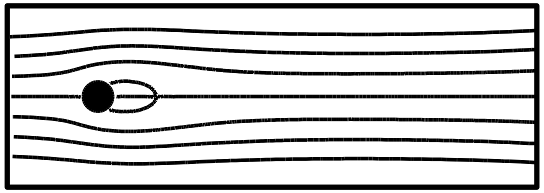
POD Galerkin model

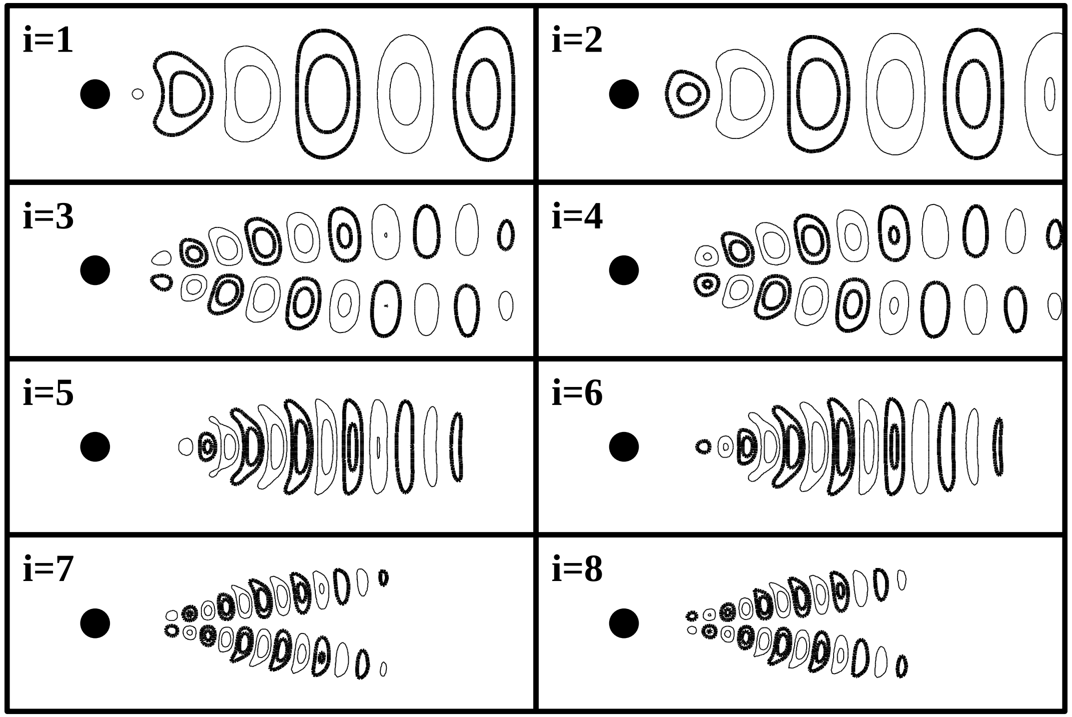
Noack, Afanasiev, Morzyński, Tadmor & Thiele (2003) JFM

POD at $Re = 100$

Deane et al (1991) PF

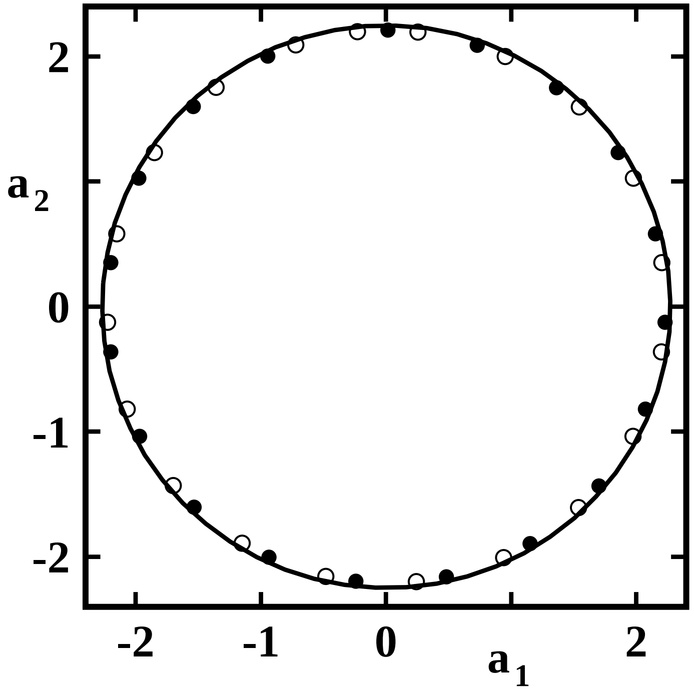
$$\mathbf{u} = \sum_{i=0}^{8} a_i \mathbf{u}_i$$





Galerkin solution

$$\frac{da_i}{dt} = \nu \sum_j l_{ij} a_j + \sum_{j,k} q_{ij} a_j a_k$$



- 8-dim. POD model reproduces DNS.

POD Galerkin system of cylinder wake

— ≡ *Noack, Afanasiev, Morzyński, Thiele & Tadmor 2003 JFM* —

Dynamical system:

$$
\begin{aligned}
da_1/dt &= \sigma a_1 - \omega a_2 + h_1 \\
da_2/dt &= \sigma a_2 + \omega a_1 + h_2 \\
da_3/dt &= \sigma_2 a_3 - 2\omega a_4 + h_3 \\
da_4/dt &= \sigma_2 a_4 + 2\omega a_3 + h_4 \\
da_5/dt &= \sigma_3 a_5 - 3\omega a_6 + h_5 \\
da_6/dt &= \sigma_3 a_6 + 3\omega a_5 + h_6 \\
da_7/dt &= \sigma_4 a_7 - 4\omega a_8 + h_7 \\
da_8/dt &= \sigma_4 a_8 + 4\omega a_7 + h_8 \\
h_i &= \sum_{j=1}^{8} \sum_{k=1}^{8} q_{ijk} a_j a_k
\end{aligned}
$$

Modal energy: $E_i = \overline{a_i^2}/2$

$$
\begin{aligned}
0 &= 2\sigma E_1 + T_1 \\
0 &= 2\sigma E_2 + T_2 \\
0 &= 2\sigma_2 E_3 + T_3 \\
0 &= 2\sigma_2 E_4 + T_4 \\
0 &= 2\sigma_3 E_5 + T_5 \\
0 &= 2\sigma_3 E_6 + T_6 \\
0 &= 2\sigma_4 E_7 + T_7 \\
0 &= 2\sigma_4 E_8 + T_8 \\
T_i &= \sum_{j=1}^{8} \sum_{k=1}^{8} q_{ijk} \overline{a_i a_j a_k}
\end{aligned}
$$

- **Dynamical system = harmonically related oscillators with growth rates** $\sigma > 0 > \sigma_2 > \sigma_3 > \sigma_4$.
- **Quadratic coupling is energy preserving:** $\sum T_i = 0$

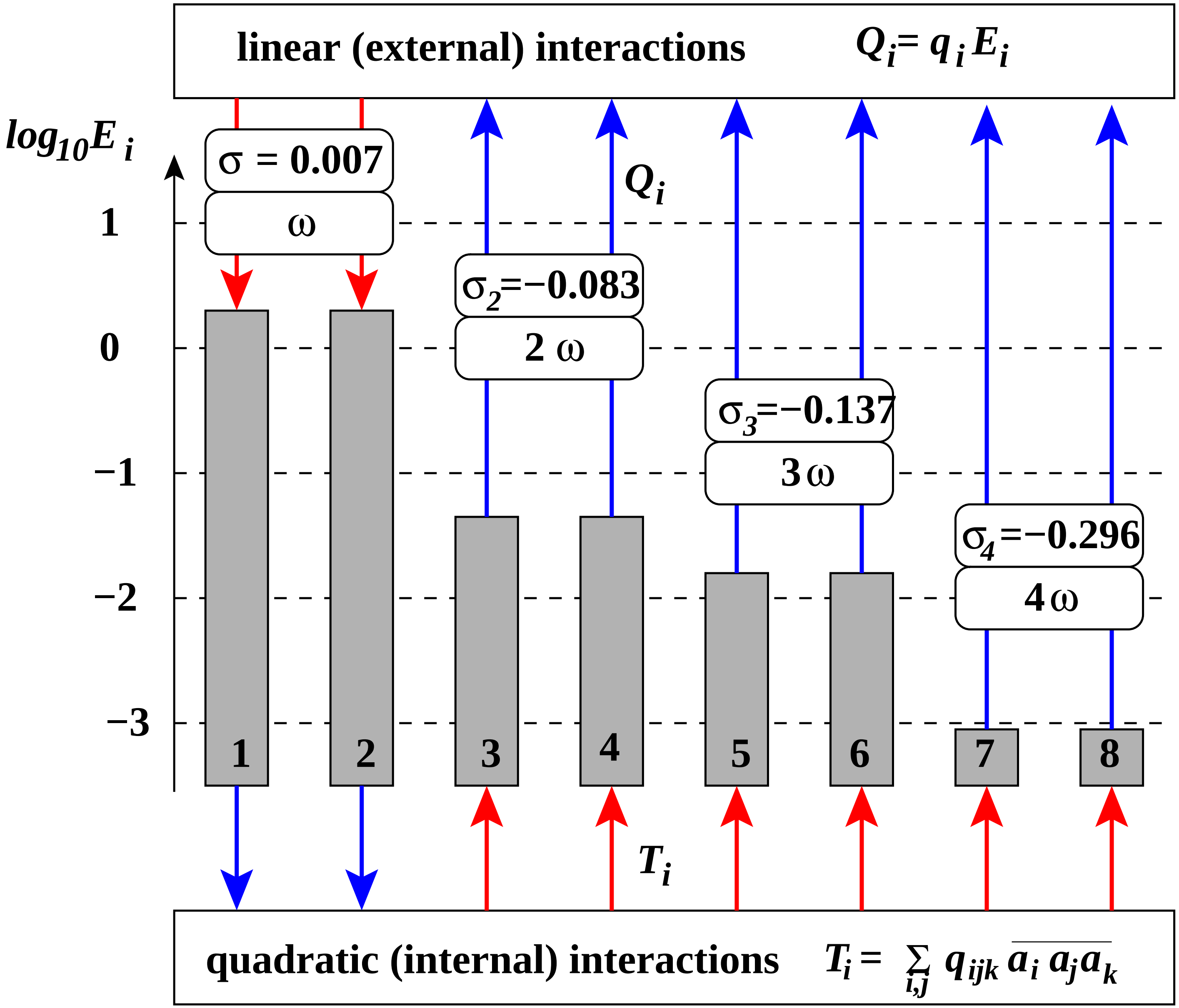
POD Galerkin system of cylinder wake II

— ≡ *Noack, Afanasiev, Morzyński, Thiele & Tadmor 2003 JFM* —

Modal energetics:

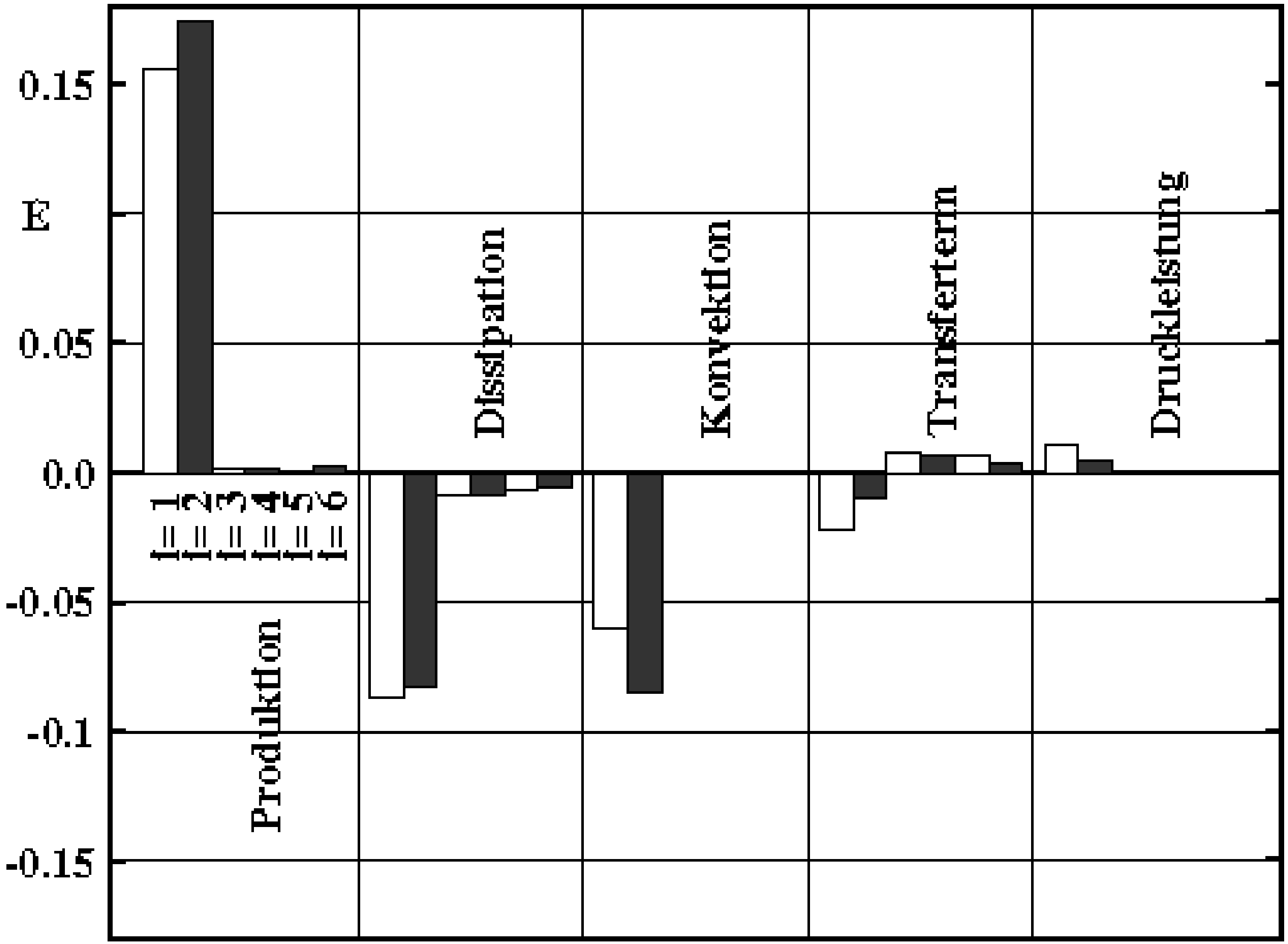
$0 = Q_i + T_i$

$Q_i = q_i E_i$



Modal energy flow analysis of cylinder wake

— ≡ *Noack 2006* —

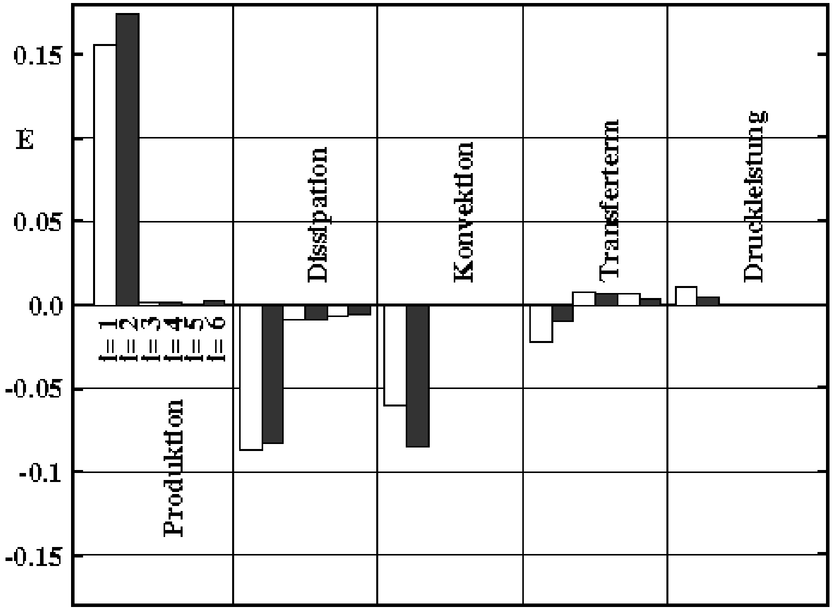


Semi-spectral characterization of e-flow cascade

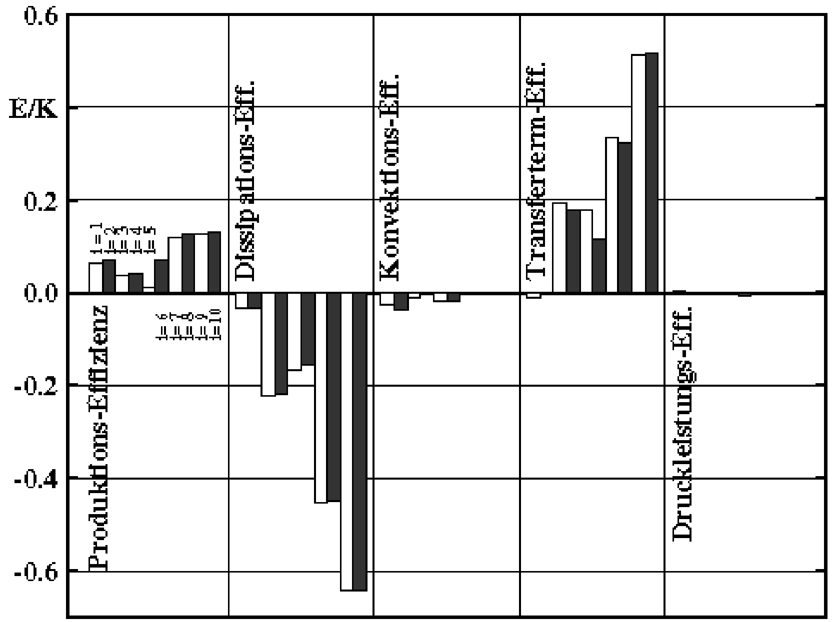
Modal energy flow analysis of cylinder wake

— Noack 2006 —

Modal energy flows



Modal e-flow efficiencies



- Semi-spectral characterization of e-flow cascade
- Implications for accuracy

Overview

1. Cylinder wake (standard method)

2. Laminar shear layer (+pressure model)

3. Turbulent mixing layer (+turbulence model)

4. Take home messages

Motivation

POD Galerkin method

$$
\begin{array}{ccccccccc}
\mathbf{u} & \rightarrow & \partial_t \mathbf{u} & = \nu \triangle \mathbf{u} & & -\nabla(\mathbf{u}\mathbf{u}) & & -\nabla p \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\mathbf{u} = \sum_{i=0}^{N} a_i \mathbf{u}_i & \rightarrow & \frac{da_i}{dt} & = \nu \sum_{j=0}^{N} l_{ij} a_j & + & \sum_{j,k=0}^{N} q_{ijk}\, a_j\, a_k & & +???
\end{array}
$$

Pressure-term modelling $-(\mathbf{u}_i, \nabla p)$

(1) $-(\mathbf{u}_i, \nabla p) = 0$ . . . . . [Deane et al. 1991, Holmes et al. 1998]

(2) $-(\mathbf{u}_i, \nabla p) =$ empirical force . . . . [Aubry et al. 1988]

(3) Escape with $\nabla \times \nabla p = 0$ . . . . [e.g. Rempfer 1991]

$$\partial_t \omega = +\nabla \omega \cdot \mathbf{u} - \nabla \mathbf{u} \cdot \omega + \nu \triangle \omega$$

(4) Calibrate a linear term . . . . [Galletti et al. 2004]

Question: $\boxed{-(\mathbf{u}_i, \nabla p) = \mathbf{f}_i(a_1, a_2, \ldots, a_N)}$ **?**

Pressure-term representation

— Noack, Papas & Monkewitz (2005) JFM —

Poisson eq. $\quad \triangle p = s \quad \ldots\ldots\ldots\ldots\ldots \quad s = -(\nabla \mathbf{u})^t : (\nabla \mathbf{u})$

$\downarrow \qquad \uparrow \qquad \downarrow \quad \mathbf{u} = \sum a_i \mathbf{u}_i$

Solution $\quad p = \sum_{j,k=0}^{N} p_{jk}\, a_j\, a_k \leftarrow \quad s = \sum_{j,k=0}^{N} s_{jk}\, a_j\, a_k$

$\triangle p_{jk} = s_{jk} \qquad s_{jk} = -(\nabla \mathbf{u}_j)^t : (\nabla \mathbf{u}_k)$

$\downarrow \qquad \downarrow$

Galerkin projection

$$\frac{d}{dt} a_i = \ldots - (\mathbf{u}_i, \nabla p)_\Omega$$

$$= \ldots - \left(\mathbf{u}_i, \nabla \left[\sum_{j,k=0}^{N} p_{jk}\, a_j\, a_k \right] \right)_\Omega$$

$$= \nu \sum_{j=0}^{N} l_{ij} a_j + \sum_{j,k=0}^{N} (q_{ijk} + q^\pi_{ijk}) a_j a_k$$

where $q^\pi_{ijk} = \left(\mathbf{u}_i, \nabla p_{jk} \right)_\Omega$

Key enablers: treatment of BC and numerical algorithm!

POD of Kelvin-Helmholtz vortices

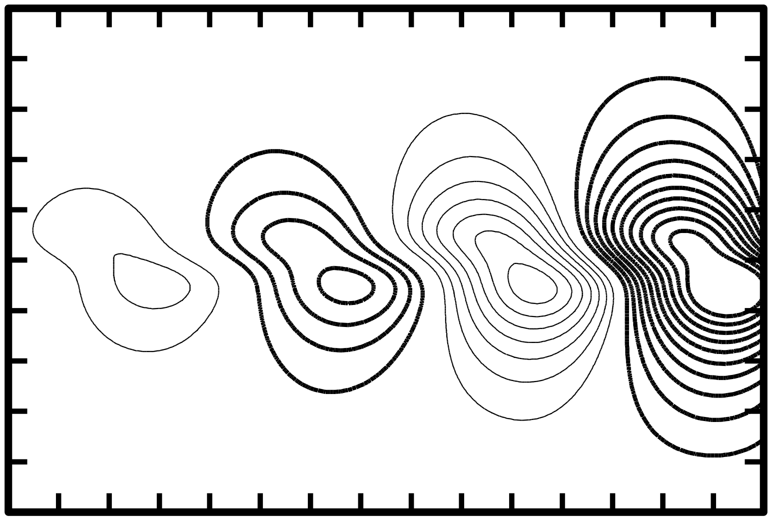
— Noack, Papas & Monkewitz (2005) JFM —

DNS

$Re_c = 100$

$u = \frac{2}{3} + \frac{1}{3} \tanh y$

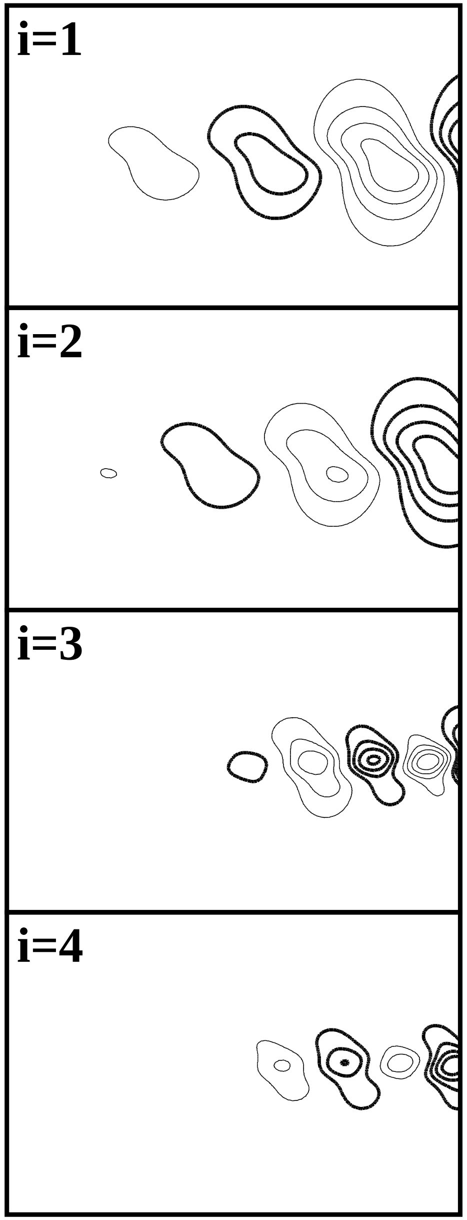
- velocity ratio 3:1
- Dirichlet inflow condition
  (+0.01 × eigenmode)
- conv. outflow condition



POD

$\mathbf{u} = \sum_{i=0}^{4} a_i \mathbf{u}_i$

- mode 1
  $\sim \sin x$
- mode 2
  $\sim \cos x$
- mode 3
  $\sim \sin 2x$
- mode 4
  $\sim \cos 2x$



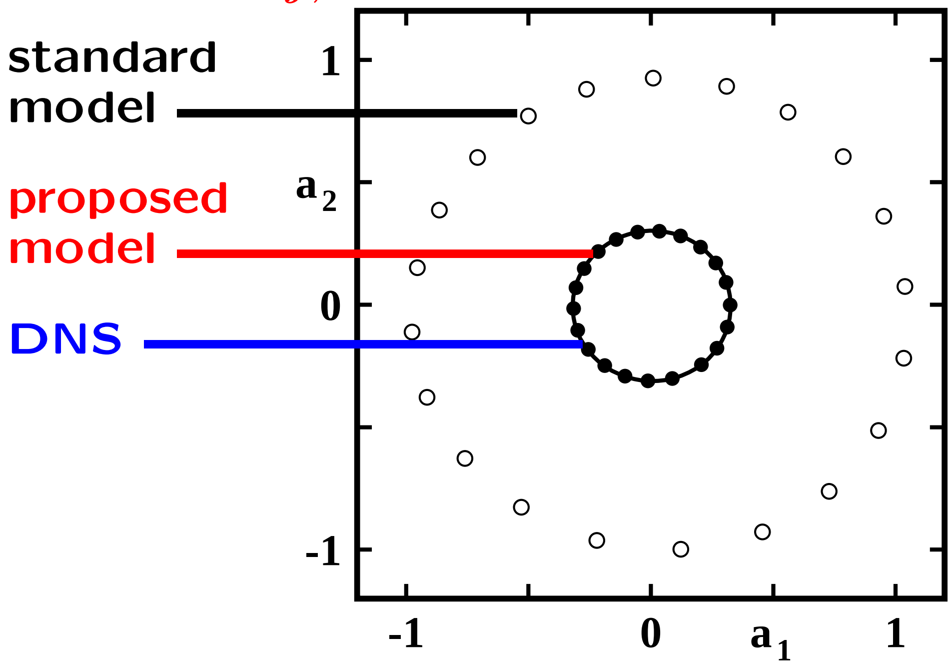
POD Galerkin model of K-H vortices

— Noack, Papas & Monkewitz (2005) JFM —

Galerkin solution

$$\dot{a}_i = \nu \sum_{j=0}^{4} l_{ij} a_j + \sum_{j,k=0}^{4} q_{ijk} a_j a_k$$

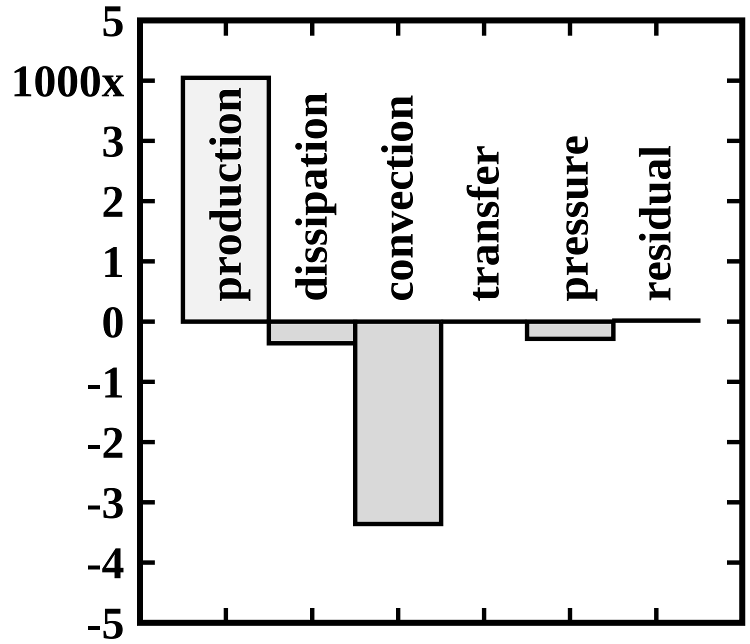
$$+ \sum_{j,k=0}^{4} q^{\pi}_{ijk} a_j a_k$$



Energy flow balance

$$0 = \frac{d}{dt}\mathcal{K} = \mathcal{P} + \mathcal{D} + \mathcal{C} + \mathcal{T} + \mathcal{F}$$

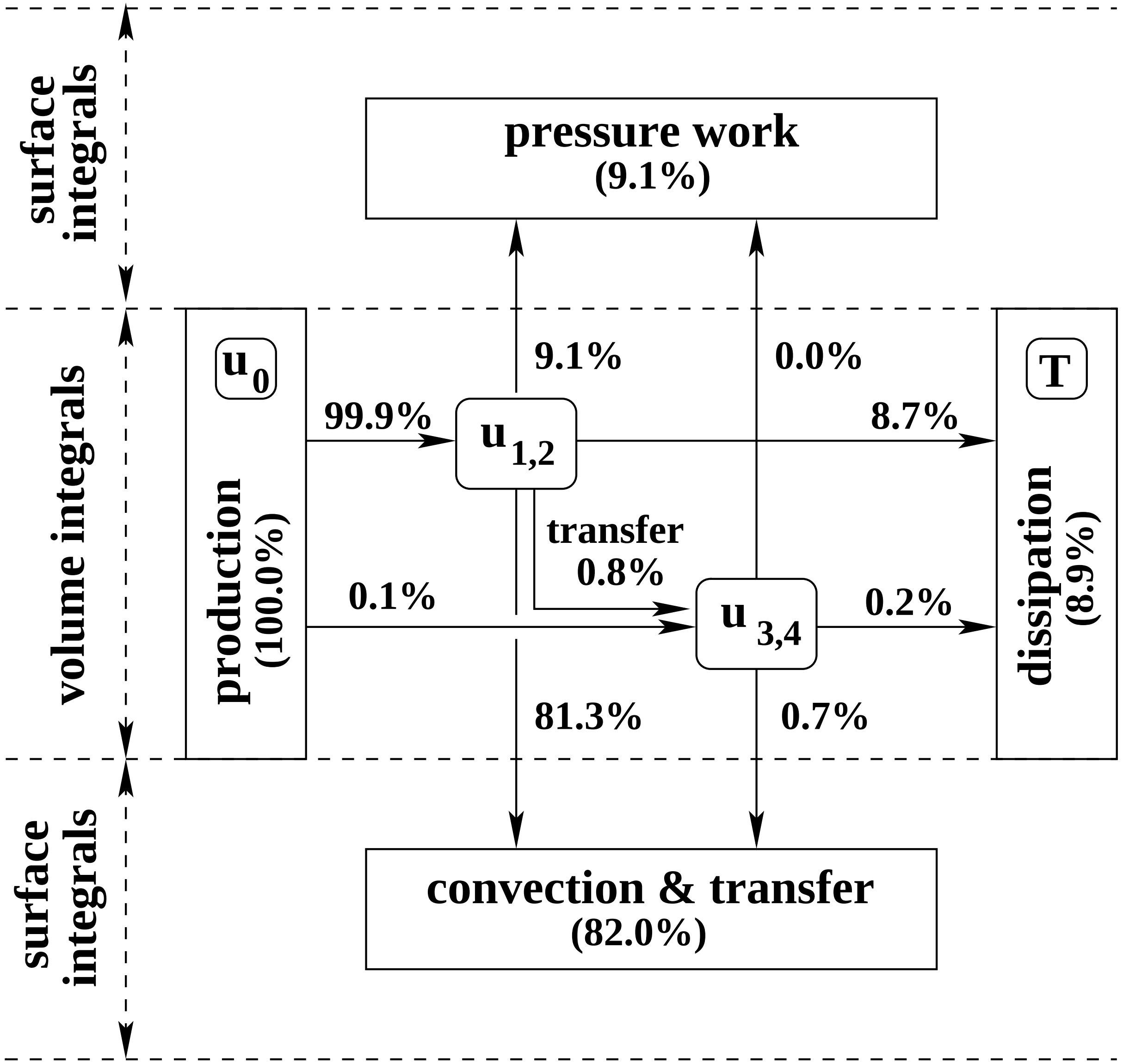
$$\mathcal{F} = \langle (\mathbf{u}', -\nabla p)_\Omega \rangle = -\oint d\mathbf{A} \cdot \langle \mathbf{u}' p \rangle$$



The pressure term must be included in the POD model!

Modal energy flow cascade of K-H vortices

— ≡ *Noack, Papas & Monkewitz (2005) JFM* —



Social wellfare.

Large modes

work for

the poor!

Pressure models — analytical to brute force

— ≡ *Noack, Papas & Monkewitz (2005) JFM* —

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(1) Analytical model . . . . $q^{+}_{ijk} = (\mathbf{u}_i, -\nabla p_{jk})_\Omega$

Ansatz: $p(\mathbf{x},t) = \sum_{j,k} p_{jk}(\mathbf{x})\, a_j(t)\, a_k(t)$ where $\triangle p_{jk} = \rho_{jk}$

- *requires pressure Poisson solver!*

(2) Empirical model . . . . $l^{+}_{ij} = (\mathbf{u}_i, -\nabla p_j)_\Omega$

Ansatz: $p'(\mathbf{x},t) = \sum_{j} p_j(\mathbf{x})\, a_j(t)$ with p_j from linear fit

- *requires (only) pressure data!*

(3) Least-order model . $\frac{d}{dt}a_i = \alpha_i a_i + \nu \sum_j l_{ij} a_j + \sum_{j,k} q_{ijk} a_j a_k$

α_i are obtained from a modal energy flow balance with a_i of the Navier-Stokes attractor

- *requires neither a pressure solver nor pressure data!*

Overview

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1. Cylinder wake (standard method)

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4. Take home messages

Motivation

POD Galerkin method

$$\begin{array}{ccccccc} \mathbf{u} & \to & \partial_t \mathbf{u} & = \nu \triangle \mathbf{u} & & -\nabla(\mathbf{u}\mathbf{u}) & -\nabla p \\ \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \\ \mathbf{u} = \sum_{i=0}^{N} a_i \mathbf{u}_i & \to & \frac{da_i}{dt} & = \nu \sum_{j=0}^{N} l_{ij} a_j & + & \sum_{j,k=0}^{N} (q_{ijk} & + q_{ijk}^{\pi}) \, a_j \, a_k \end{array}$$

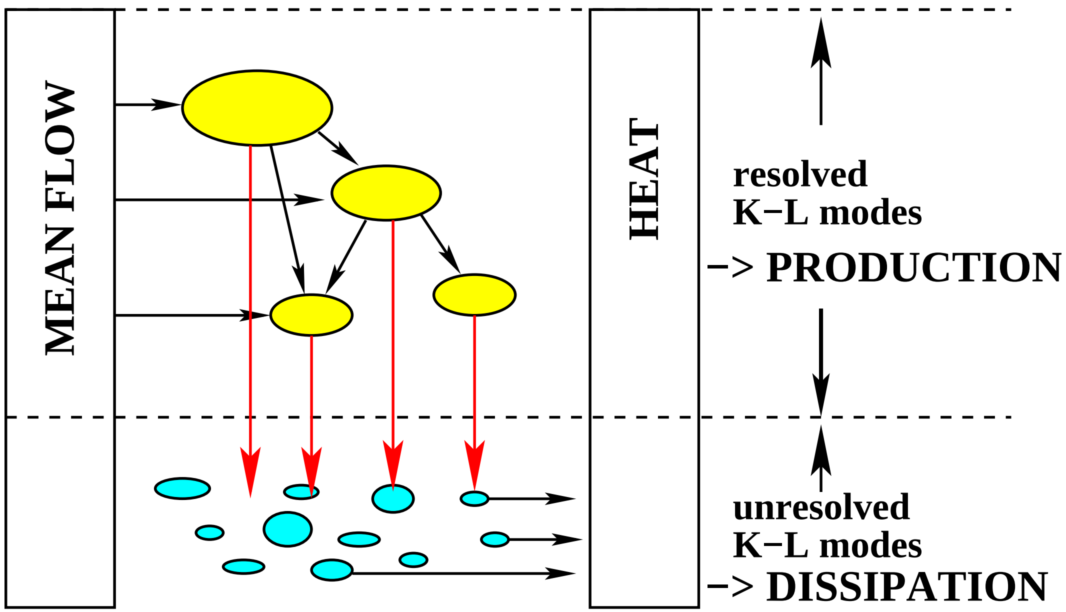
$$+ \sum_{i=N+1}^{\infty} a_i \mathbf{u}_i \qquad + \sum_{j=0}^{N} \nu_{T,i} \, l_{ij} a_j$$

Rempfer 1991

Modelling problem:

For $N \sim 10$, a typical situation is

$$\| \sum_{i=0}^{N} a_i \mathbf{u}_i \| < \| \sum_{i=N+1}^{\infty} a_i \mathbf{u}_i \|.$$



'Subgrid' turbulence modelling problem

POD Galerkin method

$$
\begin{array}{ccccc}
\mathbf{u} & \rightarrow \partial_t \mathbf{u} & = \nu \triangle \mathbf{u} & -\nabla(\mathbf{u}\mathbf{u}) & -\nabla p \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbf{u} = \sum_{i=0}^{N} a_i \mathbf{u}_i & \rightarrow \frac{da_i}{dt} & = \nu \sum_{j=0}^{N} l_{ij} a_j & + \sum_{j,k=0}^{N} (q_{ijk} & + q^{\pi}_{ijk}) \, a_j \, a_k \\
+\mathbf{u}_{\text{residual}} & & + \sum_{j=0}^{N} l^{+}_{ij} a_j & + \sum_{j,k=0}^{N} q^{+}_{ijk} a_j a_k &
\end{array}
$$

Decision 1: 'Subgrid'-turbulence models

(1) $l^{+}_{ij} = \nu_T \, l_{ij}$ (1 parameter) . . . . [Aubry et al. 1988]

(2) $l^{+}_{ij} = \nu_{T,i} \, l_{ij}$ (N parameters) . . . . [Rempfer 1991]

(3) l^{+}_{ij}-calibration (N^2 parameters) . . . [Galletti+ 2004 etc.]

(4) quadratic and cubic terms (myriad parameters)

...

Decision 2: What is a good calibration technique?

Calibration techniques

Given: $\mathbf{u}^{\mathsf{DNS}}(\mathbf{x},t) := \sum_{i=0}^{N} a_i^{\mathsf{DNS}}(t)\,\mathbf{u}_i(\mathbf{x}) + \mathbf{u}_{\mathsf{residual}}(\mathbf{x},t)$

Wanted: $\frac{a_i^{\mathsf{GM}}}{dt} = f_i = \sum_{j=0}^{N}(\nu+\nu_{T,i})\,l_{ij}\,a_j^{\mathsf{GM}} + \sum_{j,k=0}^{N} q_{ijk}\,a_j^{\mathsf{GM}}\,a_k^{\mathsf{GM}}$

with $\boxed{\mathbf{a}_i^{\mathsf{GM}} \approx \mathbf{a}_i^{\mathsf{DNS}}}$

- **Floquet calibration** .... solution matching

$$\chi_0(\{\nu_{T,i}\}_{i=1}^{N}) := \sum_{i=1}^{N}\int_0^T dt\,\left[a_i^{GM}(t) - a_i^{DNS}(t)\right]^2 = \mathsf{Min}$$

- **Poincaré calibration** .... phase space matching

$$\chi_1(\{\nu_{T,i}\}_{i=1}^{N}) := \sum_{i=1}^{N}\int_0^T dt\,\left[\dot{a}_i^{DNS}(t) - f_i\left(\mathbf{a}^{DNS}(t)\right)\right]^2 = \mathsf{Min}$$

- **Energy-flow calibration** .... e-flow consistency

$$\chi_2(\{\nu_{T,i}\}_{i=1}^{N}) := \sum_{i=1}^{N}\left[P_i + C_i + D_i + T_i + F_i\right]^2 = \mathsf{Min}$$

where P_i represents the modal production, ...

Modal fluid dynamics

— ≡ *Noack, Papas & Monkewitz (2005) JFM* —

In a nutshell:

Galerkin approximation . $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}', \quad \mathbf{u}_0 := \overline{\mathbf{u}}, \quad \mathbf{u}' := \sum_{i=1}^{N} a_i \mathbf{u}_i$

Navier-Stokes Eq. . . . . $\mathcal{R}(\mathbf{u}) = 0$

Galerkin system . . . . $\left(\mathbf{u}_i, \mathcal{R}(\mathbf{u}^{[N]})\right)_\Omega = 0$

Modal energy flow balance $\left(a_i \mathbf{u}_i, \mathcal{R}(\mathbf{u}^{[N]})\right)_\Omega = 0$

Global energy flow balance $\left(\mathbf{u}', \mathcal{R}(\mathbf{u}^{[N]})\right)_\Omega = 0$

$$\overline{F} = \frac{1}{T}\int_0^T dt\, F$$

$$(\mathbf{u},\mathbf{v})_\Omega := \int_\Omega dV\, \mathbf{u}\cdot\mathbf{v}$$

Im some detail:

NSE	NSE II	GS	modal E	
$\partial_t \mathbf{u} =$	$\partial_t \mathbf{u}' =$	$da_i/dt =$	$\frac{d}{dt}\overline{a_i^2}/2 =$	$d\, K_i /dt =$
$-\nabla\cdot\mathbf{u}\mathbf{u}$	$-\nabla\cdot\mathbf{u}_0\mathbf{u}_0$	$+q_{i00}$		
	$-\nabla\cdot\mathbf{u}'\mathbf{u}_0$	$+\sum_{j=1}^N q_{ij0}a_j$	$+2q_{ii0}\, K_i$	$+\mathcal{P}_i$
	$-\nabla\cdot\mathbf{u}_0\mathbf{u}'$	$+\sum_{j=1}^N q_{i0j}a_j$	$+2q_{i0i}\, K_i$	$+\mathcal{C}_i$
	$-\nabla\cdot\mathbf{u}'\mathbf{u}'$	$+\sum_{j,k=1}^N q_{ijk}a_j a_k$	$+\sum_{j,k=1}^N q_{ijk}\overline{a_i a_j a_k}$	$+\mathcal{T}_i$
$+\nu\triangle\mathbf{u}$	$+\nu\triangle\mathbf{u}_0$	$+\nu l_{i0}$		
	$+\nu\triangle\mathbf{u}'$	$+\nu\sum_{j=1}^N l_{ij}a_j$	$+2\nu l_{ii}\, K_i$	$+\mathcal{D}_i$
$-\nabla p$	$-\nabla p$	$+\sum_{j,k=1}^N q_{ijk}^\pi a_j a_k$	$+\sum_{j,k=1}^N q_{ijk}^\pi \overline{a_i a_j a_k}$	$+\mathcal{F}_i$

LES of turbulent mixing layer

— ≡ *Comte, Sivestrini & Bégou (1998) EJMB*—

LES of

mixing

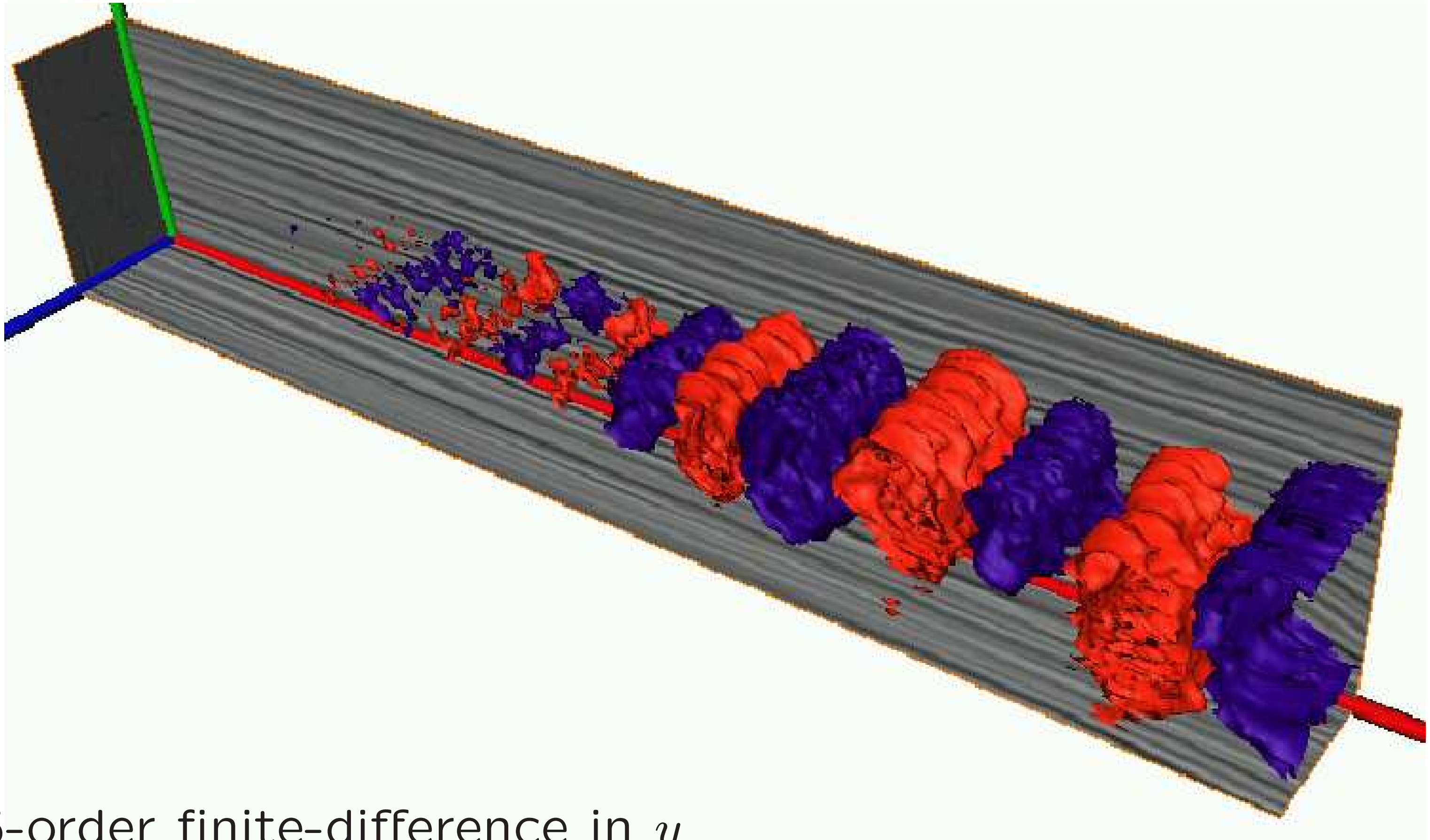
layer

at $Re \to \infty$

$U_1/U_2 = 3$

Visualization:

$v = \pm 0.04$



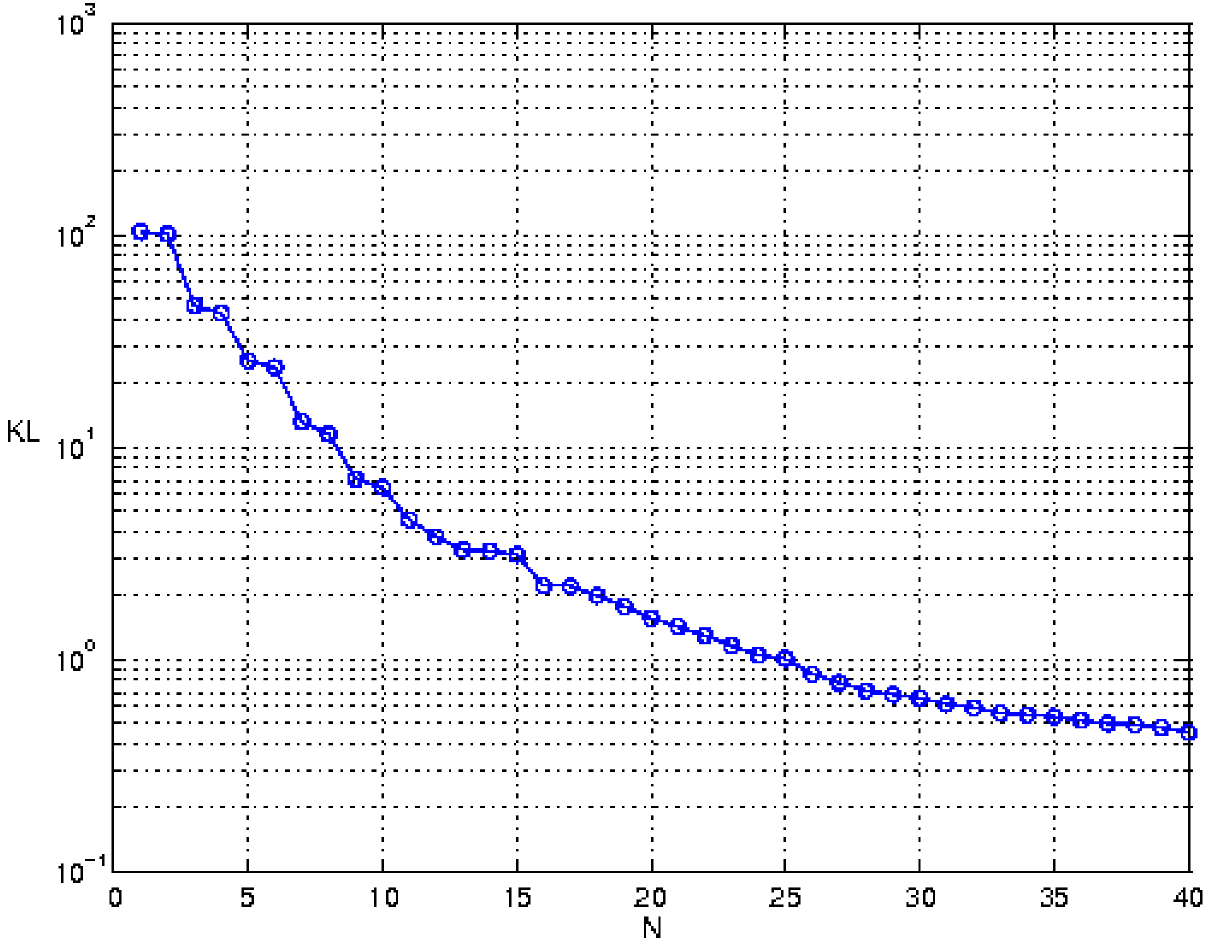
spectral in x,z, 6-order finite-difference in y

$0 \leq x/\delta_{sl} \leq 140$; $-14 \leq y/\delta_{sl} \leq 14$; $0 \leq z/\delta_{sl} \leq 15$

POD

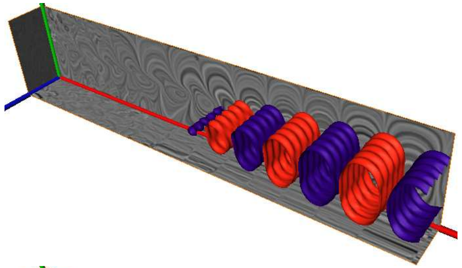
— ☰ *Noack, Pelivan, Comte, Morzyński & Tadmor (2004)* —

POD spectrum

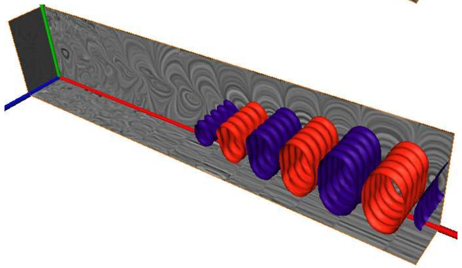


POD modes $\mathbf{u}_i$

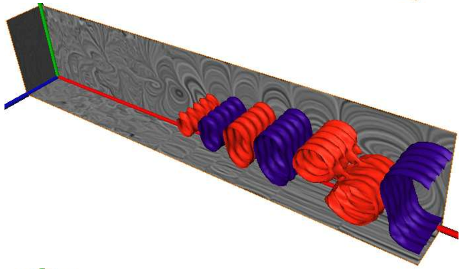
i=1 *[22%]*



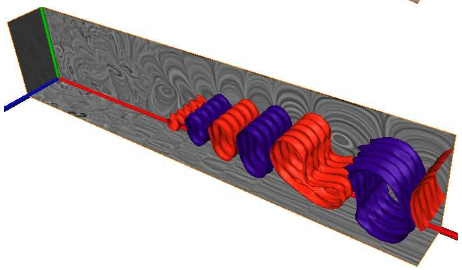
i=2 *[21%]*



i=3 *[10%]*



i=4 *[9%]*

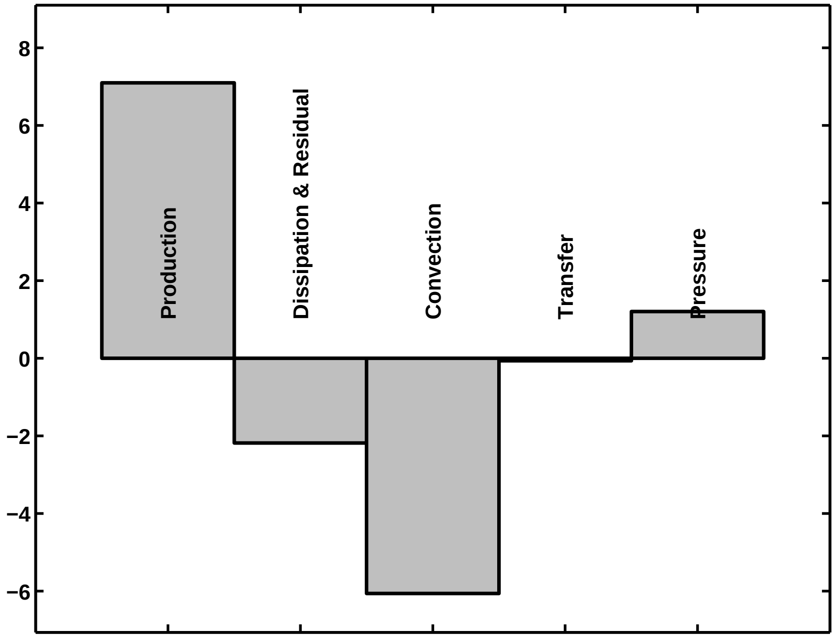


Energy flow analysis

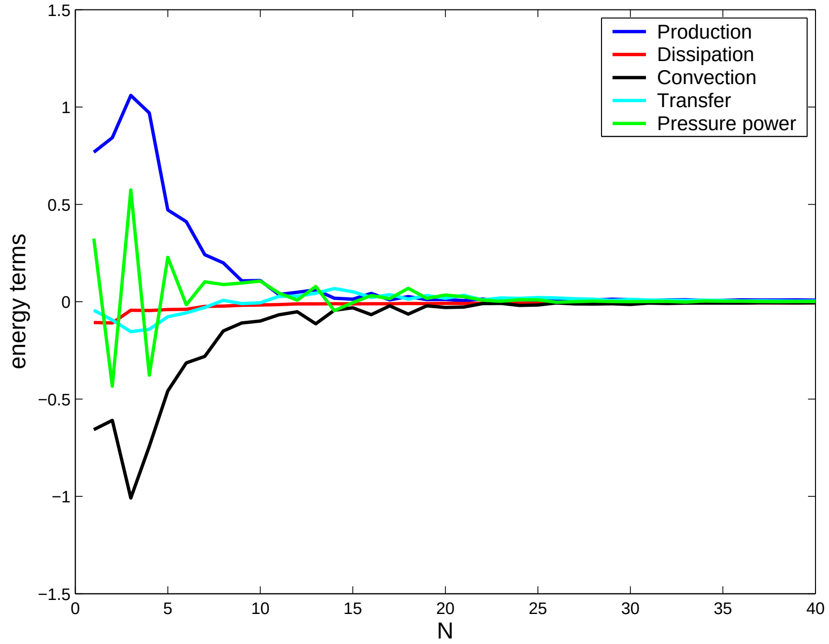
— Noack, Tadmor & Morzyński (2004) AIAA —

Global analysis	$dK/dt=$	$P+$	$D+$	$C+$	$T+$	F	$=0$
$\uparrow \sum_{i=1}^{N}$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	
Modal analysis	$dK_i/dt=$	P_i+	D_i+	C_i+	T_i+	F_i	$=0$

Global analysis



Modal analysis



Energy flow calibration

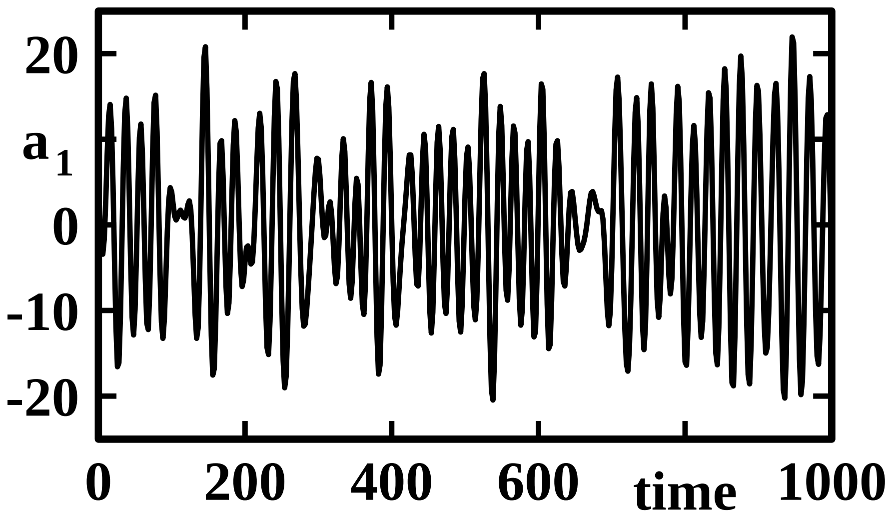
— ≡ *Noack, Pelivan, Comte, Morzyński & Tadmor (2004)* —

GM with 20 POD modes and modal eddy viscosities

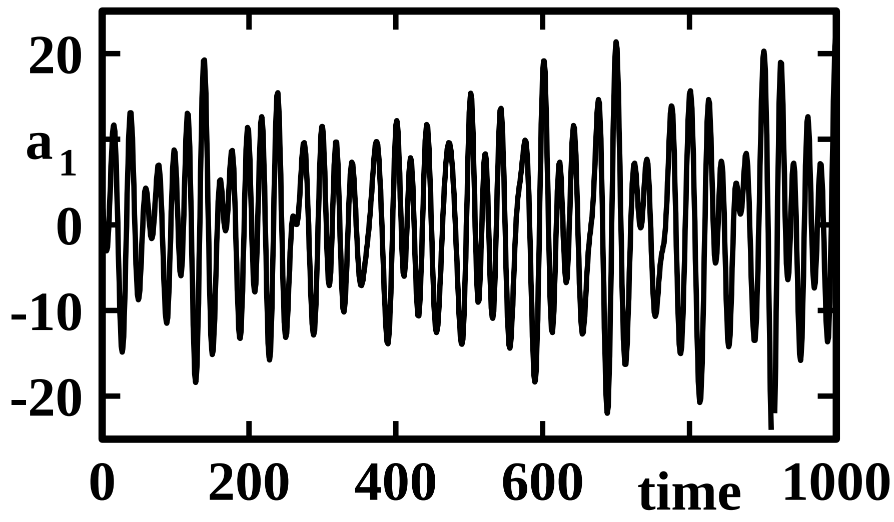
$$P_i + C_i + \left(1 + \frac{\nu_{T,i}}{\nu}\right) D_i + T_i + F_i = 0$$

— *Pars pro toto, the first Fourier coeffient* a_1 *is shown* —

LES



Galerkin model



Calibrated GM matches statistics and frequency spectra

Poincaré calibration

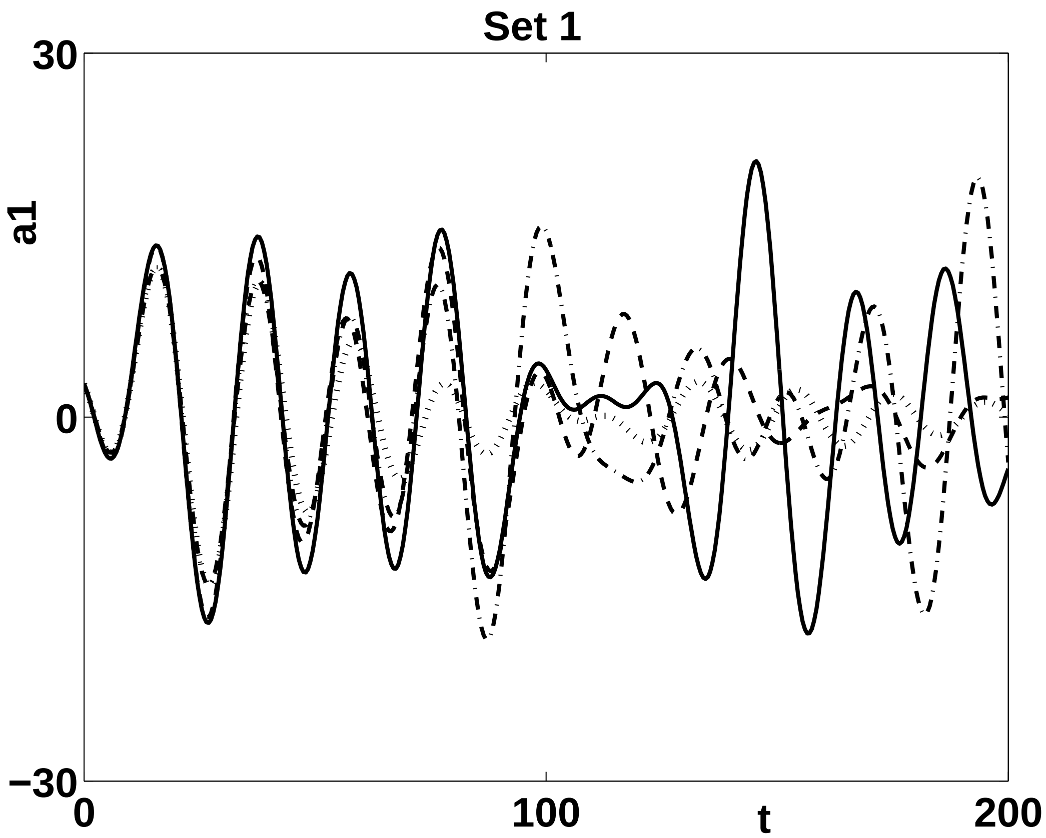
— ≡ *Tadmor & Noack (2004) ACC* —

Linear and quadratic Galerkin system with 20 POD modes

$$\chi_1 = \frac{1}{2}\int_0^T dt \sum_{i=1}^{N} \left[\dot{a}_i^{LES} - f_i(a_i^{LES})\right]^2 = \text{Min}$$

First Fourier coefficient

——	LES	
. . . .	GS with $\nu_{T,i}$	N par.
. - .	GS with l_{ij}^{+}	$\sim N^2$ par.
- - -	GS with l_{ij}^{+}, q_{ijk}^{+}	$\sim N^3$ par.



Small benefits by adding more terms.

Floquet calibration

— ☰ *Pelivan, Noack, Comte & Cordier (2005)* —

GM with 16 POD modes + modal eddy viscosity

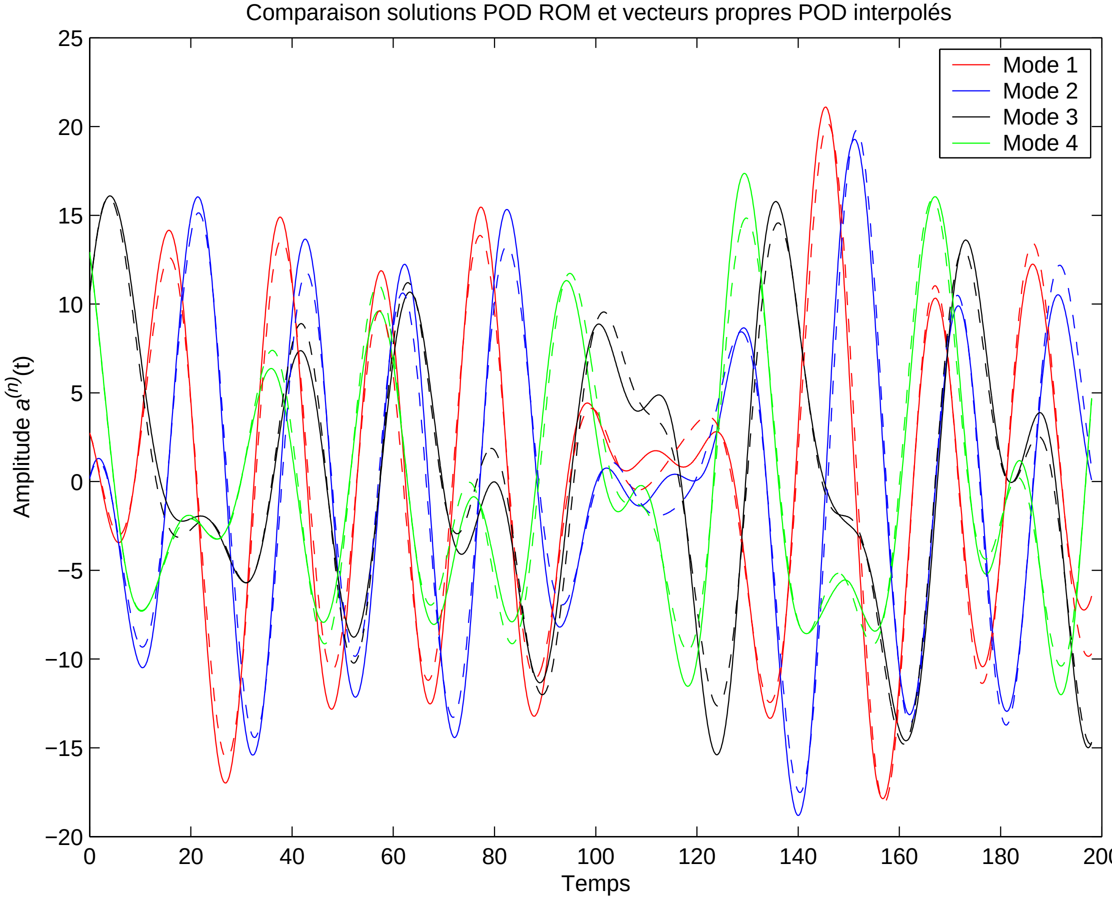
$$\chi_0 = \frac{1}{2}\int_0^T dt \sum_{i=1}^{N} \left[a_i^{GM} - a_i^{LES}\right]^2 + \frac{\beta}{2}\int_0^T dt \sum_{i=1}^{N} \left[\frac{\nu_{T,i}}{\nu}\right]^2 = \text{Min}$$

First four

Fourier coefficients

— LES

- - GM



Floquet calibration has longer prediction horizon!

Conclusions

$$\begin{array}{ccccc} \mathbf{u} & \rightarrow \partial_t \mathbf{u} & = \nu \triangle \mathbf{u} & - \nabla (\mathbf{u}\mathbf{u}) & - \nabla p \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{u} = \sum_{i=0}^{N} a_i \mathbf{u}_i & \rightarrow \frac{da_i}{dt} & = \nu \sum_{j=0}^{N} l_{ij} a_j & + \sum_{j,k=0}^{N} (q_{ijk} & + q^{\pi}_{ijk}) \, a_j \, a_k \\ + \mathbf{u}_{\text{residual}} & & + \nu_{T,i} \, l_{ij} a_j & & \end{array}$$

Floquet calibration $\chi_0 := \sum_{i=1}^{N} \int dt \, \left[a_i^{GM} - a_i^{NS} \right]^2 = \text{Min}$

$+$ largest prediction horizon $[0, T]$ for given initial condition

$-$ No limit $T \rightarrow \infty$ (phase error accumulation).

Poincare calibration $\chi_1 := \sum_{i=1}^{N} \int dt \, \left[\dot{a}_i^{NS} - f_i(\mathbf{a}^{NS}) \right]^2 = \text{Min}$

- limit $T \rightarrow \infty$ possible, independence of time window

E-flow calibr. $\chi_2 := \sum_{i=1}^{N} [P_i + C_i + D_i + T_i + F_i]^2 = \text{Min}$

- comparable to Poincaré calibration.

Overview

1. Cylinder wake (standard method)

2. Laminar shear layer (+pressure model)

3. Turbulent mixing layer (+turbulence model)

4. Take home messages

Summary

$$\mathbf{u} \rightarrow \partial_t \mathbf{u} = \nu \triangle \mathbf{u} - \nabla(\mathbf{u}\mathbf{u}) - \nabla p$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\mathbf{u} = \sum_{i=0}^{N} a_i \mathbf{u}_i \rightarrow \frac{da_i}{dt} = \nu \sum_{j=0}^{N} l_{ij} a_j + \sum_{j,k=0}^{N} \left(q_{ijk} + q_{ijk}^{\pi} \right) a_j \, a_k$$

$$+\mathbf{u}_{\text{residual}} \qquad +\nu_{T,i} \, l_{ij} a_j$$

(1) Perform standard POD method.

(2) Perform modal energy flow analysis.

(3) Add pressure term

if indicated by energy flow analysis.

(4) Add subgrid turbulence model

if POD resolves fraction of total fluctuation energy.